On the Dulac's Problem for Tangential Polycycles

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Abstract

The study of cycles for piecewise smooth systems has several open problems and interesting phenomena can happen even in the simplest cases: considering two open regions separated by a straight line. Thus, many questions that have already been solved for smooth systems can be rewritten and extended to this kind o system. In this work we propose an extension of the Dulac's problem in this new context.

Introduction and Statement of the Problem

The well known Hilbert's 16^{th} problem gave rise to a lot of works and it has been a motivation to many researchers until now. A first step towards the solution of this problem is to prove that a polynomial vector field on $\mathbb{R}^2$ has at most a finite number of limit cycles. This finiteness question can be reduced to the problem of non-accumulation of limit cycles for a polynomial vector field, called *Dulac's Problem*:

- *A elementary polycycle of an analytic vector field X cannot have limit cycles accumulating onto it.*

An elementary polycycle is a closed oriented curve formed by a finite union of regular orbits and elementary singular points of X. The main objective of this work is putting together the Dulac's problem and the study of typical tangential polycycles for piecewise analytic vector fields, a similar approach can be found in [1]. We consider planar piecewise analytic systems presenting connections between tangential singularities and investigate a version of the Dulac's Problem for this kind of tangential polycycles.

Consider a smooth embedded submanifold $\Sigma = h^{-1}(0) \subset \mathbb{R}^2$ where $h : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function for which 0 is a regular value. In this way, Σ splits $\mathbb{R}^2$ in two open regions $\Sigma^+ = \{p \in \mathbb{R}^2; h(p) > 0\}$ and $\Sigma^- = \{p \in \mathbb{R}^2; h(p) < 0\}$. A piecewise analytic vector field in $\mathbb{R}^2$ is a vector field of the form

$$Z(p) = \begin{cases} X(p), p \in \Sigma^+, \\ Y(p), p \in \Sigma^-, \end{cases}$$

where X and Y are analytic vector fields in $\mathbb{R}^2$. Denote by Ω^ω the set of all piecewise analytic vector fields defined as above. In Σ the following regions are distinguished (see Figure 1)

- Crossing region: $\Sigma^c = \{p \in \Sigma;\ Xh \cdot Yh(p) > 0\}$,
- Sliding region: $\Sigma^s = \{p \in \Sigma;\ Xh(p) < 0 \text{ and } Yh(p) > 0\}$,
- Escaping region: $\Sigma^e = \{p \in \Sigma;\ Xh(p) > 0 \text{ and } Yh(p) < 0\}$,

where $Xh(p) = \langle X, \nabla h\rangle(p)$ is the Lie derivative of h, at p, in the direction of X, analogously for $Yh(p)$. Trajectories of Z follow the Filippov convention, see [3].

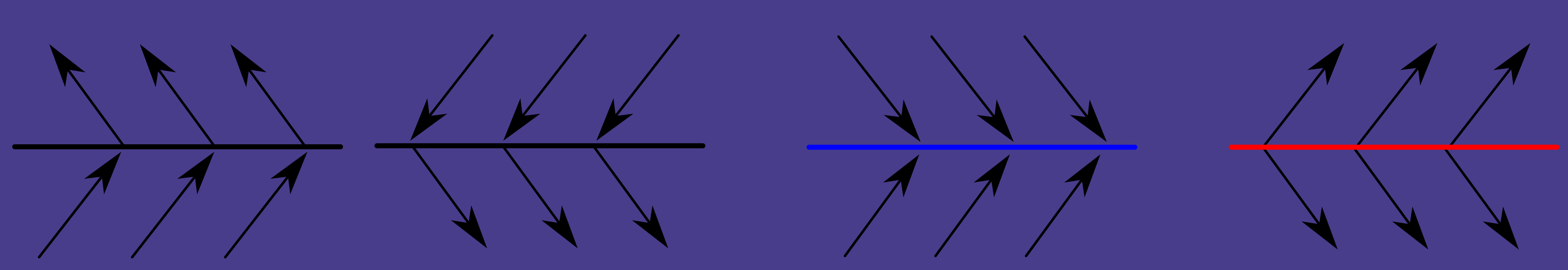

Figure 1: Regions defined in Σ: (a-b) Σ^c; (c) Σ^s; (d) Σ^e.

Consider $Z = (X, Y) \in \Omega^\omega$, a point $p \in \Sigma$ is said to be a tangential singularity of order k (k a positive integer) of X (resp. of Y) if $Xh(p) = \cdots = X^{k-1}h(p) = 0$ and $X^k h(p) \neq 0$ (resp. $Yh(p) = \cdots = Y^{k-1}h(p) = 0$ and $Y^k h(p) \neq 0$). A polycycle is said to be a tangential polycycle if all its singularities are tangential singularities (tangential-regular or tangential-tangential), see Figure 2.

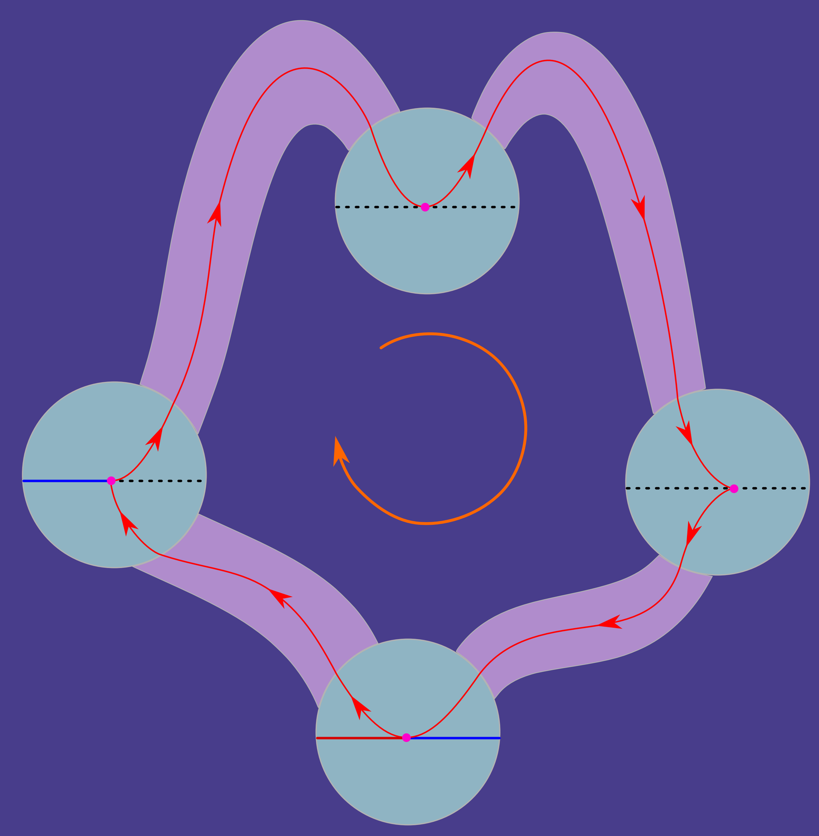

Figure 2: Example of a tangantial polycycle.

Main Result: Dulac's Problem for Tangential Polycycles

In this work, [2], we prove the following result:

Theorem A. *A tangential polycycle of a piecewise analytic vector field $Z = (X, Y) \in \Omega^\omega$ cannot have limit cycles accumulating onto it.*

Proof. The proof of this fact follows the same ideas used in the classical problem, [4], and in the [1]. Consider the first return map defined near the tangential polycycle Γ.

$$P(x) = R_n \circ D_n \circ \cdots \circ R_1 \circ D_1(x). \quad (1)$$

The proof consists in showing that the equation $P(x) - x = 0$ does not admit solutions accumulating on 0. As seen in [1] it is enough showing that each transition map is a strongly quasi-analytic map.

Definition: *A germ of a map $f : [0, A) \to \mathbb{R}$ is quasi-analytic if*

1. *f is quasi-regular:*
   i. *f has a representative on $[0, \delta)$ which is C^∞ on $(0, \delta)$, where δ is a positive constant;*
   ii. *f has a Dulac series (f is asymptotic to a Dulac series $\hat{f}$), i.e., a formal series $\hat{f}(x) = \sum_{i=1}^{\infty} x^{\lambda_i} P_i(\ln x)$, where $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$ is an increasing sequence of positive numbers tending to infinity, and P_i are polynomials.*
2. *f is a quasi-regular homeomorphism, i.e., f is quasi-regular and $P_1 \equiv A$, where A is some positive constant;*
3. *the map $x \to f \circ \exp(-x)$ has a bounded holomorphic extension $F(z)$ in some domain Δ_b of $\mathbb{C}$, defined by $\Delta_b = \{z = x + iy;\ x > b(1 + y^2)^{1/4}\}$ where b is a positive real number.*

The result follows by proving that the transition map near a tangential singularity of order k, under some conditions, is given by $D(x) = \alpha x^k + \mathcal{O}_{k+1}(x)$.

Conclusion

Many classical problems for smooth/analytical vector fields can have an extension for piecewise defined vector fields, then there are a lot of open interesting problems in this research area. It is worthwhile mentioning that piecewise smooth vector field are widely used to model problems in applied sciences as engineering, biology, control theory, and physics.

References

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