An Explicit Family of U_m-numbers

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Abstract

In this work, the authors generalize a previous result, extending LeVeque's method in order to prove that a certain subset of Liouville numbers applied to a rational function over $\overline{\mathbb{Q}}(x)$ gives Mahler's $U-$numbers of a certain type.

Introduction

The genesis of transcendental number theory, took place in 1844 with Liouville's result [8] on the "bad"approximation of algebraic numbers by rationals. Using this remarkable fact, he was able to build a non-enumerable set of transcendental numbers called *Liouville numbers*. This set is composed by all real numbers ξ, such that for any positive real number ω, there exists infinitely many rational numbers p/q, $q \geq 1$, such that

$$0 < \left|\xi - \frac{p}{q}\right| < \frac{1}{q^{\omega}}.$$

The first example of a Liouville number, and consequently of a transcendental number, is the so called *Liouville constant*, defined by the convergent series

$$\ell = \sum_{n\geq 1} 10^{-n!},$$

meaning that in it's decimal expansion there are 1 's in each factorial position and 0's otherwise. In 1962, Erdös [5] proved that every nonzero real number could be written by the sum of two Liouville numbers, although of being a set of null Lebesgue measure.

In the literature, several classifications of transcendental numbers have been developed, one of them proposed by Kurt Mahler in 1932 [9]. He splited the set of transcendental numbers into three disjoint sets:$S-$,$T-$ and $U-$numbers. In a certain sense, $U-$numbers generalize the concept of Liouville numbers.

Let $\omega_n^*(\xi)$ be the supremum of the real numbers ω^* for which there exist intinitely many real algebraic numbers α of degree n satisfying,

$$0 < |\xi - \alpha| < \mathcal{H}(\alpha)^{-\omega^*-1},$$

where $\mathcal{H}(\alpha)$ (so-called the *height* of α) is the maximum of absolute values of the minimal polynomial (over $\mathbb{Z}$) of α. If $\omega_m^*(\xi) = \infty$ and $\omega_n^*(\xi) < \infty$, for all $1 \leq n < m$, the number number ξ is said to be a U_m^*-number. This definition is, in fact, the definition of Koksma's U_m^*-numbers [?], not Mahler's U_m-numbers. However, it is well known that these sets are the same. We remark that the set of Liouville numbers is precisely the set of U_1-numbers.

The first one to prove the existence of U_m-numbers for all $m \geq 1$, was LeVeque [7], considering the mth root of a convenient Liouville number ($\sqrt[m]{(3+\ell)/4}$). In 2014, the authors found explicit U_m-numbers in a more natural way: as the product of certain $m-$degree algebraic numbers and ℓ [3]. In 1972, Alniaik [1] had already proved that a much stronger fact holds for *strong Liouville numbers*, defined as follows.

Definition 1 ([7]). *An irrational δ is said to be a strong Liouville number, if for every n, there exists $N = N(n)$ such that $k > N$ implies $q_{k+1} > q_k^n$.*

More specifically, that if ξ is strong Liouville, then ξ applied to a rational function with coefficients of degree at most m, is a U_m-number. Our main result, states that such fact also holds for Liouville numbers, satisfying a particular condition on a sequence of rational numbers that approximates it, but that are not necessarely strong Liouville.

Main Theorem

Theorem 1. *Let $\vartheta : \mathbb{N} \to \mathbb{N}$, such that $\omega_n := \vartheta(n+1)/\vartheta(n) \to \infty$, as $n \to \infty$. Let $\xi \in \mathbb{R}$, such that there exists an infinite sequence of rational numbers $(p_n/q_n)_n$, satisfying*

$$\left|\xi - \frac{p_n}{q_n}\right| < H\left(\frac{p_n}{q_n}\right)^{-\vartheta(n)}, \qquad (1)$$

where $H(p_{n+1}/q_{n+1}) \leq H(p_n/q_n)^{O(\vartheta(n))}$. Now, take $a_0, \dots, a_l, b_0, \dots, b_r \in \overline{\mathbb{Q}}$, with $b_r = 1$ and $a_l \neq 0$, such that the degree of the extension $\mathbb{Q}(a_0, \dots, a_l, b_0, \dots, b_r)$ over $\mathbb{Q}$ is m. Then, for $P(z), Q(z) \in \overline{\mathbb{Q}}[z]$, given by $P(z) = a_0 + a_1 z + \cdots + a_l z^l$ and $Q(z) = b_0 + b_1 z + \cdots + b_r z^r$, $P(\xi)/Q(\xi)$ is a U_m-number.

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