Maximal Monotonicity of Bifunctions

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Abstract

In this work, we present some necessary and sufficient conditions for a monotone bifunction to be maximal monotone in the pointwise sense. Moreover, we show how this approach have been used to establish existence of solutions to a class of equilibrium problems in Banach Spaces [1]. We still observe that we recover some similar results for equilibrium problems defined on Hadamard Manifolds [2]. Finally, I emphasize that all results were developed in full fill partner with my co-authors in [1] and [2].

Introduction

Monotone bifunctions have been extensively studied since the classical equilibrium problem (EP) was introduced by Blum and Oettli's paper in 1994. The problem (EP) consists to find $\bar{x} \in D$ such that $F(\bar{x}, y) \geq 0, \forall\, y \in D$ where D is an empty subset of a Banach space X and $F : X \times X \to \overline{\mathbb{R}}$ is a bifunction.

FRAMEWORK*: In the following, X^* denotes the dual space of X and $\langle \cdot, \cdot \rangle : X \times X^* \to \mathbb{R}$ is the duality pairing. We recall that a multivalued operator $T : X \rightrightarrows X^*$ is called **Monotone** if for every $x, y \in X$ and $x^* \in T(x)$, $y^* \in T(y)$, it follows that $\langle x - y, x^* - y^* \rangle \geq 0.$ It is called **Maximal Monotone (MMO)** if it is monotone and its graph, $\mathrm{Graph}(T) = \{(x, x^*) \in X \times X^* : x^* \in T(x)\}$, is not properly included in the graph of any other monotone operator. We also consider:

- $\mathcal{C}(X^*) := \{C \subset X^* / \, C$ is a closed convex set$\}$.
- $\mathcal{S}(X) := \{s : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\} /$ either $s \equiv -\infty$ or $-\infty \notin s(X), s(0) = 0, s$ is l.s.c., convex and positively homogeneous$\}$.
- $D_F := \{x \in X; F(x, y) > -\infty, \forall\, y \in X\}$.

Def. 1: We say that $F : X \times X \to \overline{\mathbb{R}}$ is a **Monotone Bifunction (MB)** iff $F(x, y) \leq -F(y, x) \quad \forall\, x, y \in X.$ If $x, y \in D_F$, we have that

$$F(x, y) \leq -F(y, x) \iff F(x, y) + F(y, x) \leq 0.$$

Idea: We are going to associate bifunctions and operators as follows:

One: Given a bifunction $F : X \times X \to \overline{\mathbb{R}}$ we define the operator

$$A^F(x) := \{x^* \in X^* / \langle y - x, x^* \rangle \leq F(x, y)\ \forall\, y \in X\}. \tag{1}$$

Two: Given a multivalued operator $T : X \rightrightarrows X^*$ we define the bifunction

$$G_T(x, y) := \sup_{x^* \in T(x)} \langle y - x, x^* \rangle. \tag{2}$$

Remark 1: Under all definitions above, we have obtained some results:

1. $D(A^F) \subseteq D_F$ and $D_{G_T} = D(T) := \{x \in X / T(x) \neq \emptyset\}$.
2. $A^F(x)$ is a closed convex set and $G_T(x, x + \cdot) \in \mathcal{S}(X), \forall x \in X$.
3. If F is an MB, then A^F is a Monotone Operator (MO).
4. An operator T is Monotone if, and only if, the bifunction G_T is Monotone.

Def. 2: Let $F : X \times X \to \overline{\mathbb{R}}$ be an MB. We say that F is a **Maximal Monotone Bifunction (MMB)** if, and only if, A^F is a **(MMO)**.

More notations and definitions

- $\mathcal{B}(X) := \{F(\cdot, \cdot)$ is a bifunction$/ \, F(x, x + \cdot) \in \mathcal{S}(X), \forall\, x \in X\}$.
- $\mathcal{M}(X) := \{F(\cdot, \cdot)$ is a MB$/ \, F(x, \cdot)$ is l.s.c. and convex, $\forall x \in X\}$.
- $\mathcal{O}(X) := \{T(\cdot)$ is a MO $/ \, T(x) \in \mathcal{C}(X^*) \forall x \in X\}$.

Remark 2: For every operator $T : X \rightrightarrows X^*$, one has $G_T \in \mathcal{B}(X)$. Thus, we have that the mapping $\mathcal{O}(X) \ni T \longmapsto G_T \in \mathcal{B}(X)$ is well-defined.

Preliminary results

We begin this section by introducing the following one-to-one mappings:

$$\varphi : \mathcal{C}(X^*) \to \mathcal{S}(X) \qquad \psi : \mathcal{S}(X) \to \mathcal{C}(X^*)$$
$$\varphi(C) = \delta^*_C \qquad \psi(s) = \{x^* / \langle x, x^* \rangle \leq s(x) \,\forall\, x\}$$

Theorem 1: $\psi(\delta^*_C) = C, \ \forall C \in \mathcal{C}(X^*)$ and $\delta^*_{\psi(s)} = s, \ \forall s \in \mathcal{S}(X)$

By considering the sets $\mathcal{B}(X)$ and $\mathcal{O}(X)$, we take the special mappings:

$$\mathcal{B}(X) \ni F \longmapsto A^F \in \mathcal{O}(X) \qquad \mathcal{O}(X) \ni T \longmapsto G_T \in \mathcal{B}(X)$$

We also have that the mappings defined above are one-to-one, which one is the inverse each other. Moreover, we proved that they comute:

Proposition 1: $A^{G_T} = T \quad \forall T \in \mathcal{O}(X), \quad G_{A^F} = F \quad \forall F \in \mathcal{B}(X).$

Next, we introduce a kind of maximal monotonicity of bifunctions:

Def. 3: Let $F \in \mathcal{B}(X) \cap \mathcal{M}(X)$. We say that F is **pointwise maximal** in $\mathcal{B}(X) \cap \mathcal{M}(X)$ iff, $H \in \mathcal{B}(X) \cap \mathcal{M}(X)$ with $F \leq H \implies F = H.$

Proposition 2: Let $F \in \mathcal{B}(X) \cap \mathcal{M}(X)$. F is an MMB if, and only if, F is pointwise maximal in $\mathcal{B}(X) \cap \mathcal{M}(X)$.

Def. 4: The **Fitzpatrick function of** the operator $T : X \rightrightarrows X^*$ is

$$\varphi_T(x, x^*) := \langle x, x^* \rangle - \inf_{(y, y^*) \in Graph(T)} \langle x - y, x^* - y^* \rangle.$$

The **Fitzpatrick transform of** $F(\cdot, \cdot)$ is $\Phi_F(x, x^*) := (-F(\cdot, x))^*(x^*)$.

Remark 3: If $H \in \mathcal{B}(X)$, then $\Phi_H = \varphi_{A^H}$.

Main Results

Theorem 2: Let $F \in \mathcal{M}(X)$. Then the following statements are equivalent:

1. F is MMB.
2. G_{A^F} is PMMB.
3. For every MMB bifunction $H \in \mathcal{B}(X) \cap \mathcal{M}(X)$ such that Φ_H is finite-valued, there exists $x_H \in X$ such that
$$F(x_H, y) + H(x_H, y) \geq 0 \ \forall\, y \in X.$$
4. There exist $H \in \mathcal{B}(X)$ and $p \in X$ such that:
   - $H(p, \cdot) : X \to \mathbb{R}$ is affine.
   - $H(p, z) + H(z, p) < 0, \ \forall z \in X \setminus \{p\}$.
   - For every $(x_0, x_0^*) \in X \times X^*$, there exists $\tilde{x} \in X$ satisfying
$$F(\tilde{x}, y) + H(x_0 - \tilde{x}, x_0 - y) - \langle y - \tilde{x}, x_0^* \rangle \ \geq 0 \ \ \forall\, y \in X.$$

Proposition 3: Let $A : X \to X^*$ be an MO and $B : X \to X^*$ be an MMO such that φ_B is finite-valued. Consider the following statements:

1. A is a MMO. 2. For every $x \in X$, it holds that $R(A + B(\cdot - x)) = X^*$
3. For every $x \in X$, $x' \in X$ such that $0 \in A(x') + B(x' - x)$.

Then the following implications hold true: $1 \Rightarrow 2 \Rightarrow 3$. If, moreover, B is single valued and strictly monotone, then the three statements are equivalent.

Remark 4. i) The equivalences in the Theorem 2 state the existence of solutions to equilibrium problems obtained by perturbing the given monotone bifunction F with a pointwise monotone bifunction H. ii) The equivalence $1 \iff 3$ in the Proposition 3 generalizes the following sentence: T is an MMO if, and only if, for each $x \in X$ there exists $x' \in X$, such that $0 \in \mathcal{J}(x' - x) + T(x')$. It is a classical characterization that appears in the literature.

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Referências

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