The Graceful Game

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1 Introduction

One of the most studied graph labelings is the *graceful labeling*, so named by S. W. Golomb [2] and initially introduced by A. Rosa [3] in 1996. A *graceful labeling* of a graph G with m edges is an injective function $f: V(G) \rightarrow \{0, 1, \ldots, m\}$ such that, when each edge $uv \in E(G)$ is assigned the *(induced) label* $g(uv) = |f(u) - f(v)|$, all induced edge labels are distinct. Labeling problems are usually studied from the perspective of determining whether a given graph has a required labeling or not. An alternative perspective is to analyze labeling problems from the point of view of combinatorial games. We investigate the Graceful Game, first proposed by Tuza [4].

The *Graceful Game* is defined in the following way: Alice and Bob alternately assign a previously unused label $\phi(v) \in \{0, \ldots, m\}$ to a previously unlabeled vertex v of a given graph G. If both endpoints of an edge $uv \in E(G)$ are labeled, the *label* of uv is defined as $|\phi(u) - \phi(v)|$. A move (label assignment) is said to be *legal* if, after it, all edge labels are distinct. Alice *wins* the game if the whole graph G is gracefully labeled, and Bob *wins* if he can prevent this. In this work, we study winning strategies for Alice and Bob in complete graphs, paths, cycles, complete bipartite graphs, caterpillars, prisms, wheels, hypercubes and powers of paths [1].

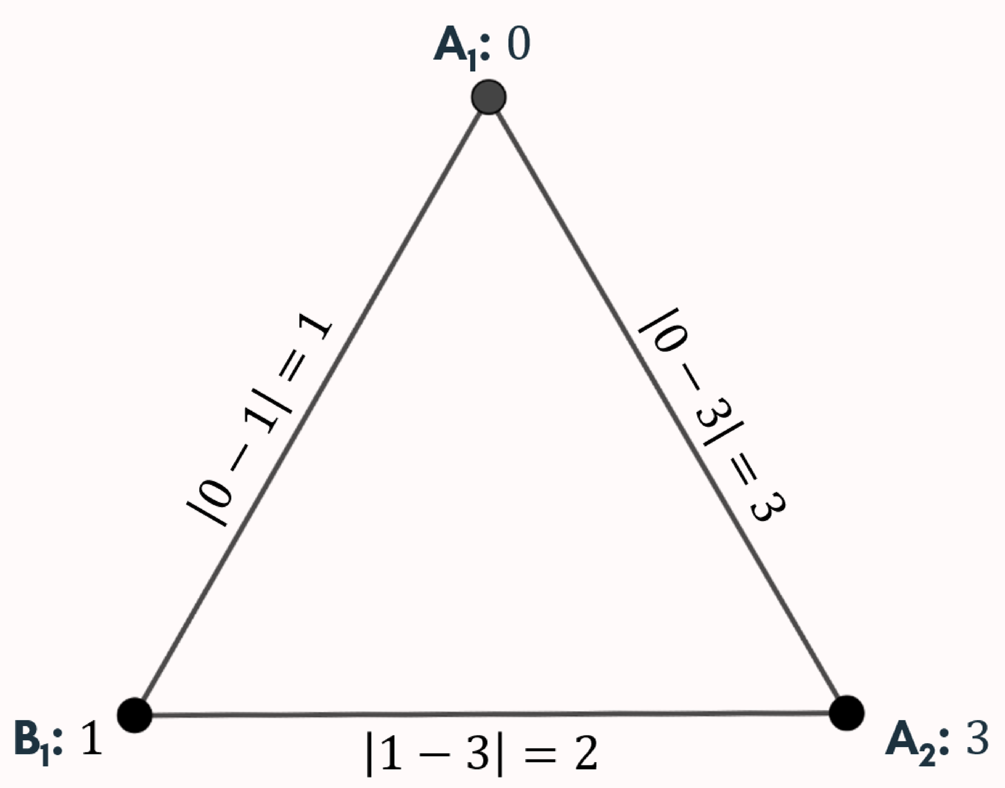

Figure 1: One of the cases for a match of the graceful game on K_3 where Alice starts.

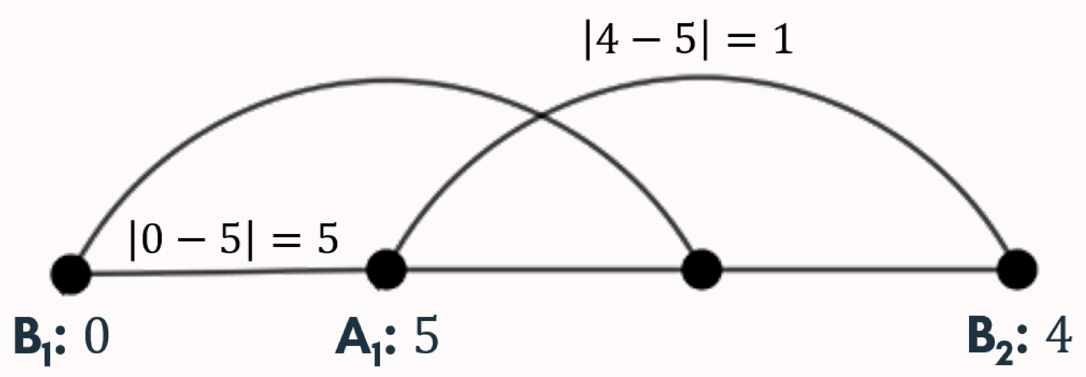

Figure 2: Bob's strategy for a case of a match on P_4^2 where he is the first player.

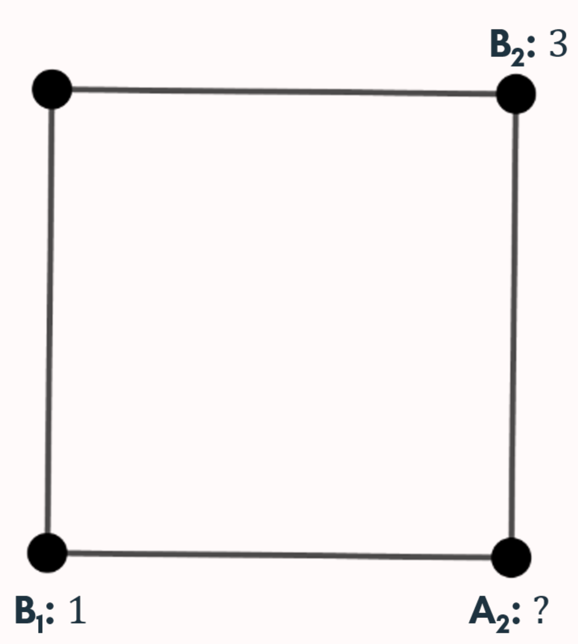

Figure 3: Bob's strategy on C_4 where here is the first player.

2 Results

CLASS OF GRAPHS \ FIRST PLAYER		ALICE STARTS	BOB STARTS
PATHS	P_1 and P_2	Alice wins	Alice wins
	P_3	Alice wins	Bob wins
	P_n, $n \geq 4$	Bob wins	Bob wins
STARS	$K_{1,n-1}$	Alice wins	Bob wins
COMPLETE GRAPHS	K_3	Alice wins	Alice wins
	K_4	Bob wins	Bob wins
CYCLES	C_3	Alice wins	Alice wins
	C_n, $n \geq 4$	Bob wins	Bob wins

Figure 4: Graceful Game results

CLASS OF GRAPHS \ FIRST PLAYER		ALICE STARTS	BOB STARTS
WHEELS	W_n, $n \geq 3$	Open	Bob wins
HELMS	H_n, $n \geq 3$	Bob wins	Bob wins
WEB GRAPHS	$W(t,n)$, $t \geq 2$, $n \geq 3$	Bob wins	Bob wins
CATERPILLAR	$cat(k_1, \ldots, k_s)$	Bob wins	Bob wins
PRISMS	$P_{r,s}$, $r \geq 8$, $s \geq 1$	Bob wins	Bob wins
HYPERCUBES	Q_k, $k \geq 2$	Bob wins	Bob wins
POWER OF PATHS	P_3^2	Alice wins	Alice wins
	P_n^2, $n \geq 4$	Bob wins	Bob wins

Figure 5: Graceful Game results

3 Conclusion

This work contributes to the advancement of the study of the graceful labeling regarding the classes of graphs approached in the article. These results were presented and published at the 17th Cologne-Twente Workshop on Graphs Combinatorial Optimization University of Twente [1].

References

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