Codimension one Principal Singularities of Hypersurfaces of $\mathbb{R}^4$

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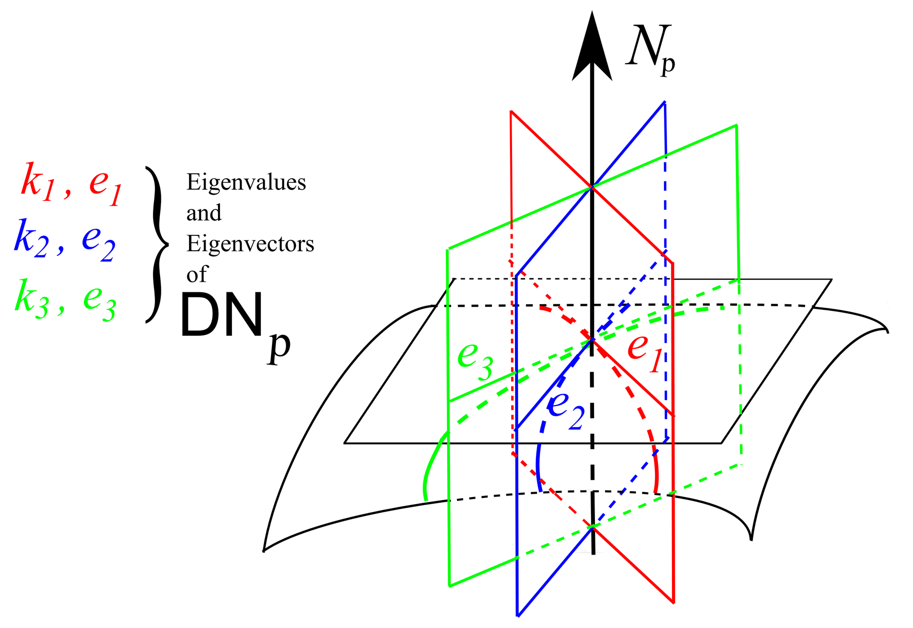
Resumo

This poster is about the mutually orthogonal one dimensional singular foliations, in oriented three dimensional manifolds $\mathbb{M}^3$, whose leaves are the integral curves of the principal curvature direction fields associated to immersions $\alpha : \mathbb{M}^3 \to \mathbb{R}^4$. We focus on behavior of these foliations around singularities defined by the points, called partially umbilic, where at least two principal curvature coincide. It be described the generic behavior of the principal foliations in the neighborhood of partially umbilic points of codimension one. These are the singularities which appear generically in one parameter families of hypersurfaces. We express the codimension one condition by minimally weakening the genericity condition given by R. Garcia, D. Lopes e J. Sotomayor in Partially Umbilic Singularities of Hypersurfaces of $\mathbb{R}^4$. Bulletin des Sciences Mathematiques (Paris. 1885), v. 139, p. 431-472, (2015)

Preliminary Results

Let M^3 be a C^∞, oriented, compact, $3-$dimensional manifold and $\alpha : M^3 \to \mathbb{R}^4$ be an immersion.

Principal Directions e_1, e_2, e_3: directions where the normal curvature $k_n(p) : T_p\mathbb{M}^3 \to \mathbb{R}$ assumes critical values;
Principal Curvature: $k_1 = k_n(p)(e_1)$, $k_2 = k_n(p)(e_2)$, $k_3 = k_n(p)(e_3)$ ($k_1 \le k_2 \le k_3$).
Principal Curvature Line: The curve whose tangent vector is a principal direction.



The *principal singularities* of the immersion α are:

- *Umbilic Points*: $\mathcal{U}_\alpha = \{p \in M^3 : k_1(p) = k_2(p) = k_3(p)\}$,
- *Partially Umbilic Points*: $\mathcal{S}_\alpha = \mathcal{S}_{12}(\alpha) \cup \mathcal{S}_{23}(\alpha)$, where $\mathcal{S}_{12}(\alpha) = \{p \in M^3 : k_1(p) = k_2(p) < k_3(p)\}$, $\mathcal{S}_{23}(\alpha) = \{p \in M^3 : k_1(p) < k_2(p) = k_3(p)\}$.

The goal is

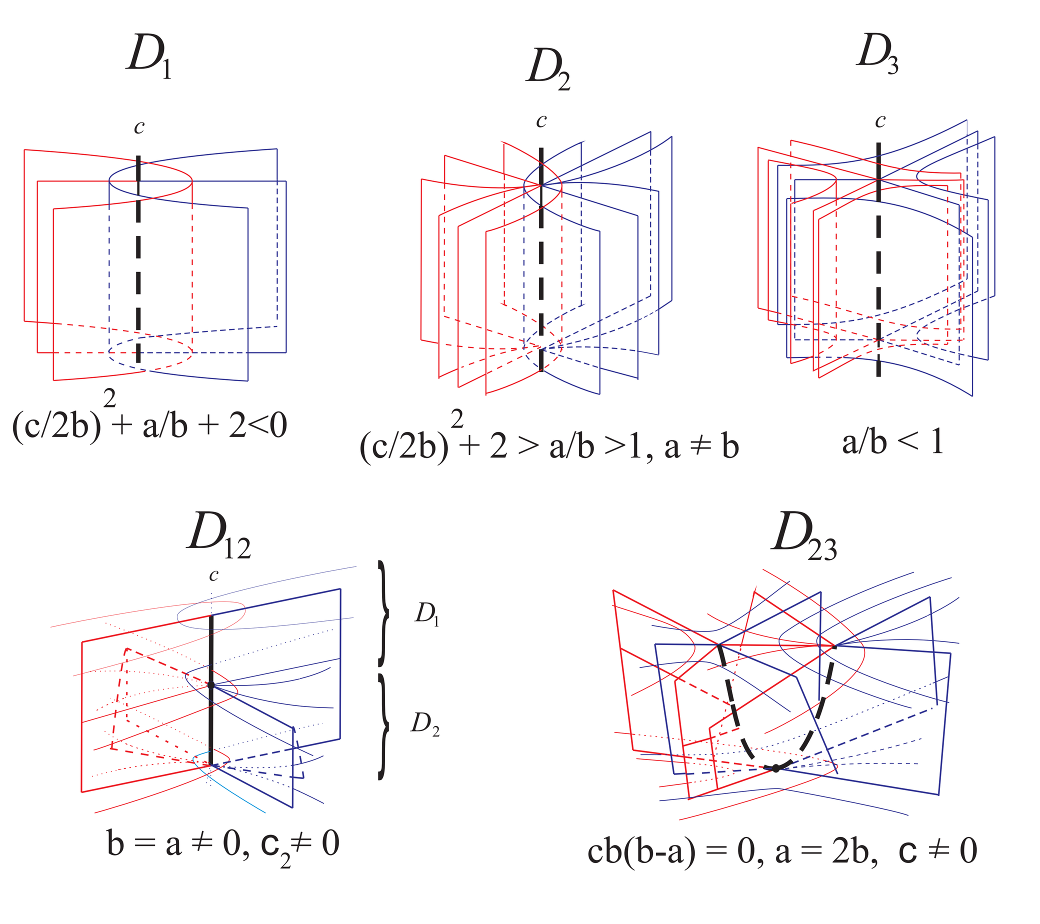
to describe the local behavior of Principal Curvature Lines

near generic Principal Singularities

in one parameter families of hypersurfaces immersed in $\mathbb{R}^4$.

The term *generic* means that the property is valid for a *collection of families which contains the intersection of countably many open dense sets* in this family. By Baire theorem a generic family is dense in the space of immersions with the C^r Whitney topology.

The generic classification of the principal singularities (see [1]) was made using the Thon Transversality Theorem. It was defined a stratification of space $\mathcal{J}(\mathcal{M}^3, \mathcal{R}^4)$ of r-jets of immersions α of $\mathbb{M}^3$ into $\mathbb{R}^4$ and it found conditions (called transversality conditions) making the jet , $j^5\alpha : \mathcal{M}^3 \to \mathbb{R}^4$, transversal to the stratification of $\mathcal{J}(\mathcal{M}^3, \mathcal{R}^4)$. This procedure leads to the five generic (cod 0) types: D_1, D_2, D_3, D_{12}, D_{23}:



If we weakening the generics conditions in the most stable and generic way, leading to codimension one ummbilic singularities.

This work has two parts:

1. In the first one, we have to find all codimension one Principal Singularities (Finding Transversality Conditions).
2. In the second one, we have to study the behavior of curvature lines in the neighborhood of all codimension one Principal Singularities (Using Lie-Cartan lifting).

Finding Transversality Conditions

A one parameter family of an immersed hypersurface will be a family $\alpha(\lambda)$ of immersions of an oriented hypersurface $\mathbb{M}^3$ into the space $\mathbb{R}^4$, where $\lambda \in \mathbb{R}$. The family α_λ will be smooth in the sense that $\alpha(m, \lambda) = \alpha_\lambda(m)$ is of class C^∞ in the product manifold $\mathbb{M}^3 \times \mathbb{R}$. The space of families, denoted by $\mathcal{F}_{\mathbb{M}^3\times\mathbb{R}}$, will be endowed with the Whitney Fine Topology

Consider the space $\mathbb{J}^r(\mathbb{M}^3, \mathbb{R}^4)$ of r-jets of immersions α of $\mathbb{M}^3$ into $\mathbb{R}^4$, endowed with the structure of Principal Fiber Bundle. The base is $\mathbb{M}^3 \times \mathbb{R}^4$, the fiber is the space $\mathcal{J}^r(3,4)$, where $\mathcal{J}^r(3,4)$ is the space of r-jets of immersions of $\mathbb{R}^3$ to $\mathbb{R}^4$, preserving the respective origins. The structure group, $\mathcal{A}^r_+$, is the product of the group $\mathcal{L}^r_+(3,3)$ of r-jets of diffeomorphisms of $\mathbb{R}^3$ preserving origin and the orientation, acting on the right by coordinate changes, and the group $\mathcal{O}_+(4,4)$ of positive isometries; the action on the left consists of a positive rotation of $\mathbb{R}^4$. Each 5-jet of an immersion at a partially umbilic point is of the form $(p, \tilde{p}, w)$ with $(p, \tilde{p}) \in \mathbb{M}^3 \times \mathbb{R}^4$ and w is in the orbit of a polynomial immersion $(u_1, u_2, u_3, h(u_1, u_2, u_3))$, where

$$\begin{aligned} h(u_1,u_2,u_3) = {} & \frac{k}{2}(u_1^2+u_2^2) + \frac{k_3}{2}u_3^2 + \\ & + \frac{a}{6}u_1^3 + \frac{b}{2}u_1u_2^2 + \frac{c}{6}u_2^3 + \\ & + \sum_{i+j+k=3;k\neq 0} q_{ijk}u_1^iu_2^ju_3^k + \\ & + \frac{A}{24}u_1^4 + \frac{B}{6}u_1^3u_2 + \frac{C}{4}u_1^2u_2^2 + \frac{D}{6}u_1u_2^3 + \frac{E}{24}u_2^4 + \\ & + \sum_{i+j+k=4;k\neq 0} Q_{ijk}u_1^iu_2^ju_3^k + \\ & + \sum_{i+j+k=5} a_{ijk}u_1^iu_2^ju_3^k + O(6) \end{aligned} \quad (1)$$

The general quadratic part of $\tilde{h}$, where $(u_1, u_2, u_3, \tilde{h})$ is in the orbit of h, has the form $k_{110}u_1u_2 + k_{101}u_1u_3 + k_{011}u_2u_3 + \frac{k_{200}}{2}u_1^2 + \frac{k_{020}}{2}u_2^2 + \frac{k_{002}}{2}u_3^2$. The manifold of partially umbilic jets, $(\mathcal{PU})^5$, is defined by the condition that the symmetric matrix

$$\begin{pmatrix} k_{200} & k_{110} & k_{101} \\ k_{110} & k_{020} & k_{011} \\ k_{101} & k_{011} & k_{002} \end{pmatrix}$$

has two equal eigenvalues, it is a submanifold of codimension 2 in $\mathcal{J}^5(3,4)$. The manifold of umbilic jets, $\mathcal{U}^5$, is defined by condition $k_{200} = k_{020} = k_{002}$ and $k_{110} = k_{101} = k_{011} = 0$. It is a closed submanifold of codimension 5 in $\mathcal{J}^5(3,4)$, that will be avoided by transversality.

TRANSVERSALITY: f is transversal to Z at $x \in M$ if:
1)$y = f(x) \notin Z$
2)$y = f(x) \in Z$ and $Df_x(T_xM) + T_yZ = T_yP$

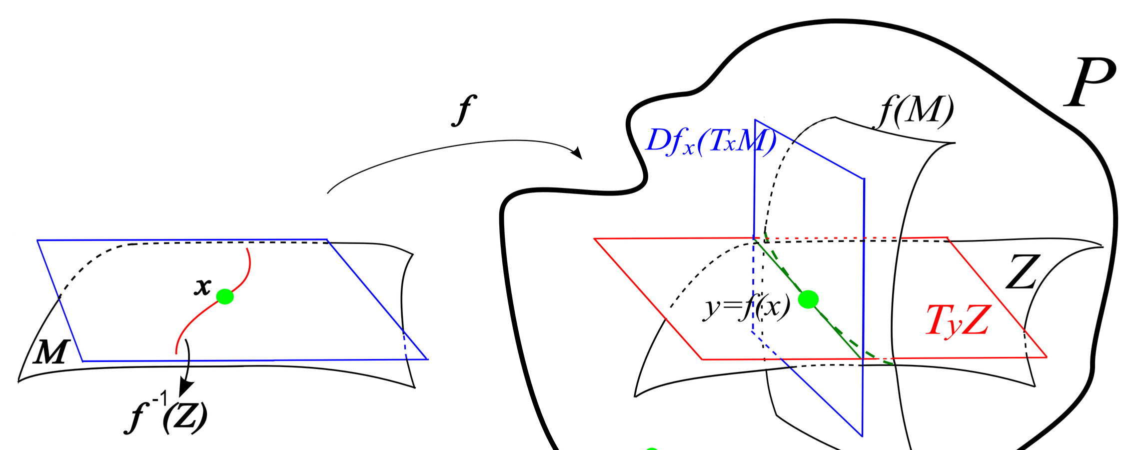


Figura 1: In the figure we do $M = \mathbb{M}^3 \times \mathbb{R}$, $P = \mathcal{J}^5(3,4)$ and $f = \alpha_\lambda$.

The Thom Transversality Theorem says that:

- $\{f \in C^k(M,P) : f$ is transversal to the $Z\}$ is residual at $C^k(M,P)$ (Dense)
- If Z has codimension n in P then $f^{-1}(Z)$ has codimension n in M (if f is transversal to Z).

$f = j^5\alpha_\lambda$:	$\mathbb{M}^3 \times \mathbb{R}$	$\to$	$J^5(3,4)$
(cod2)	$f^{-1}((\mathcal{PU})^5)$		$(\mathcal{PU})^5$(cod2)
(cod2)	$f^{-1}((\mathcal{D}_i)^5)$		$(\mathcal{D}_1)^5, (\mathcal{D}_2)^5, (\mathcal{D}_3)^5$(cod2)
(cod3)	$f^{-1}((\mathcal{D}_{12})^5), f^{-1}((\mathcal{D}_{12})^5)$		$(\mathcal{D}_{12})^5, (\mathcal{D}_{23})^5$(cod3)
(cod4)	$f^{-1}((\mathcal{D}^1_i)^5)$		$(\mathcal{D}^1_1)^5, (\mathcal{D}^1_2)^5, (\mathcal{D}^1_3)^5$(cod4)
(cod4)	$f^{-1}((\mathcal{D}^1_{13})^5), f^{-1}((\mathcal{D}^1_p)^5)$		$(\mathcal{D}^1_{13})^5, (\mathcal{D}^1_p)^5$(cod4)
(cod4)	$f^{-1}((\mathcal{D}^1_c)^5), f^{-1}((\mathcal{D}^1_{1h,p})^5)$		$(\mathcal{D}^1_c)^5, (\mathcal{D}^1_{1h,p})^5$(cod4)
(cod4)	$f^{-1}((\mathcal{D}^1_{1h,n})^5)$		$(\mathcal{D}^1_{1h,n})^5$(cod4)

Theorem 1 *Let $\mathbb{M}^3$ be a compact, oriented, smooth manifold. In the space of immersions $Imm^r(\mathbb{M}^3, \mathbb{R}^4)$, endowed with the C^6-topology, the following properties are generic for smooth one parametric families, α_λ, of immersions in $\mathcal{F}_{\mathbb{M}^3\times\mathbb{R}}$.*

i) The set $\mathcal{S}(\alpha)$ of partially umbilic points of $\alpha \in Imm^r(\mathbb{M}^3, \mathbb{R}^4)$ is empty or it is a smooth submanifold of codimension 2, a regular surface, of $\mathbb{M}^3 \times \mathbb{R}$, stratified as follows:

ii) The points D_1, D_2 and D_3 occur on regular surface.

iii) The points D_{12} and D_{23} occur along regular curves.

iii) The points D^1_1, D^1_2, D^1_3, D^1_{13}, $D^1_{1h,n}$, $D^1_{1h,p}$, D^1_p and D^1_c occur at isolated points.

Proof: Apply the Thom transversality theorem in the space of jets to make the jet extension $j^5\alpha_\lambda$, as a mapping of $\mathbb{M}^3 \times \mathbb{R}$, transversal to the canonic stratification of $\mathbb{J}^5(\mathbb{M}^3, \mathbb{R}^4)$.

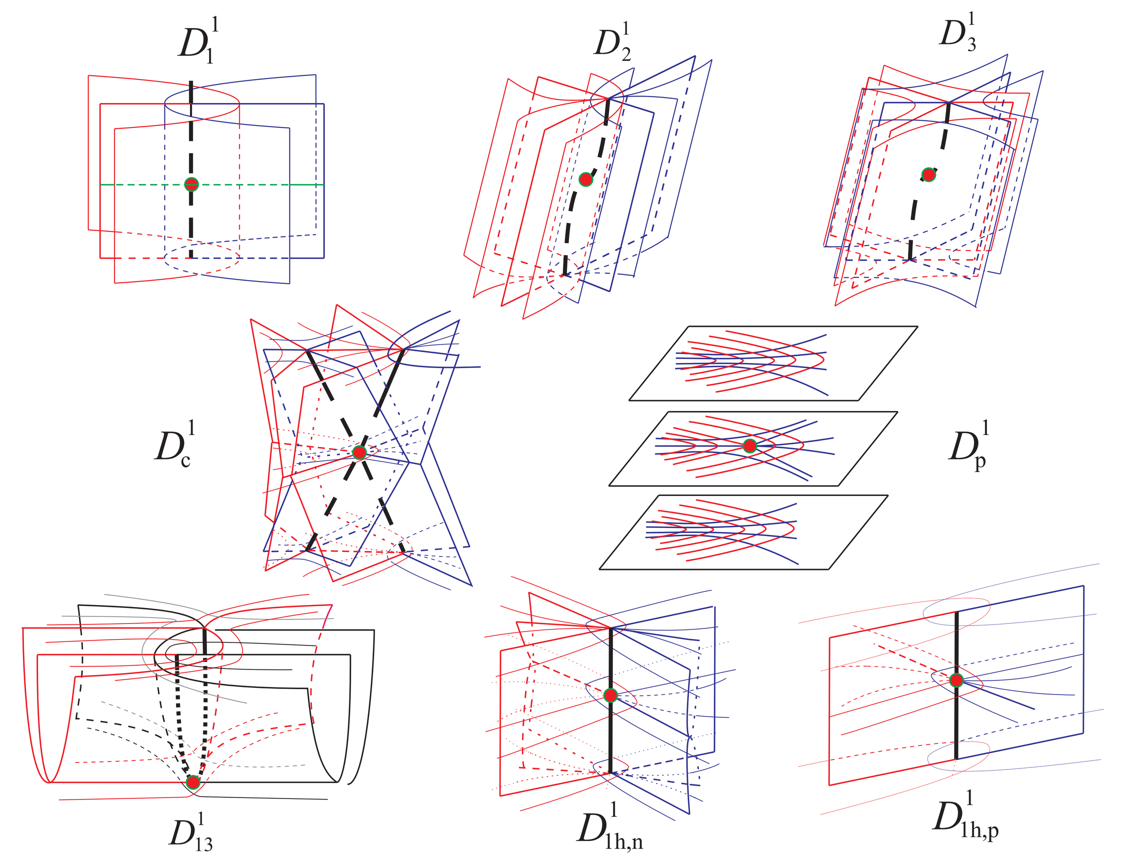
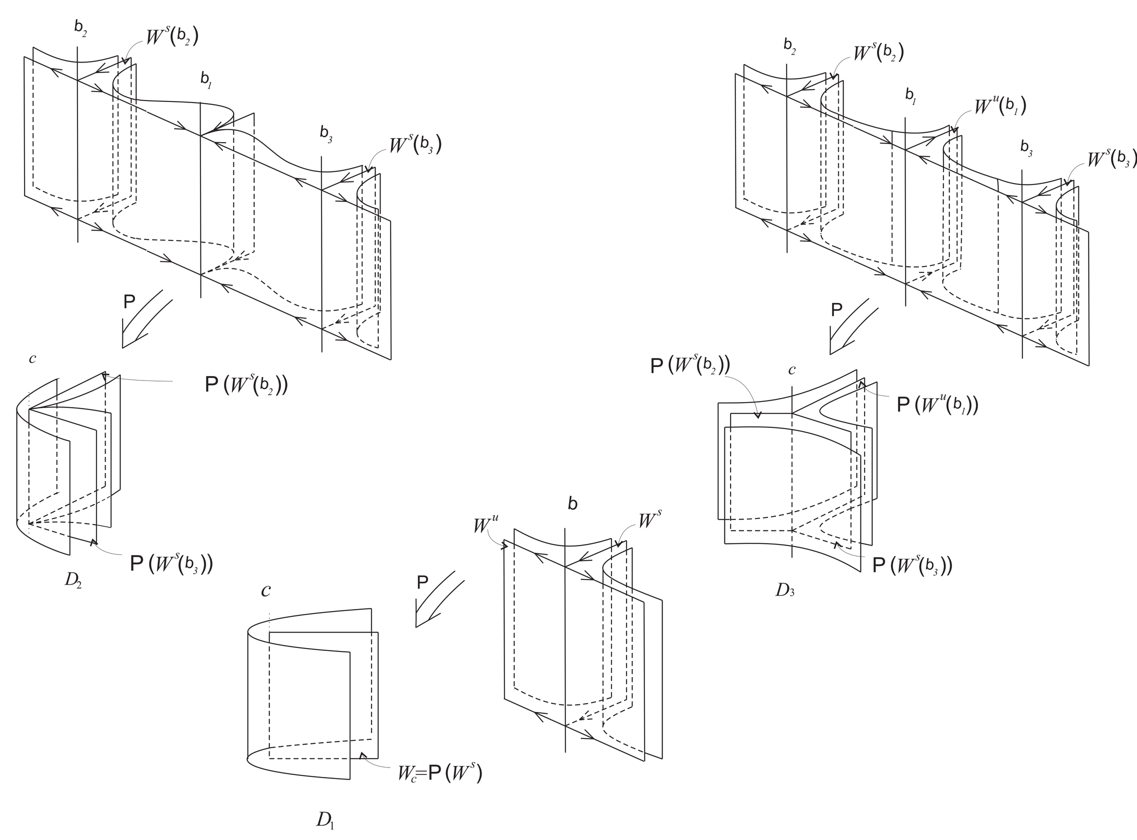


Figura 2: Codimension one Principal Singularities

Using Lie-Cartan vector Field to study the behavior of curvature lines in the neighborhood of a Partially Umbilic Point

We analyse a vector field in the projective space and whose projections of the integral curves are the principal lines of the two principal foliations $\mathcal{F}_1(\alpha)$ and $\mathcal{F}_2(\alpha)$ which are singular along the partially umbilic curve $\mathcal{S}_{12}(\alpha)$.

Lifting of Lie-Cartan: D_1, D_2, D_3.



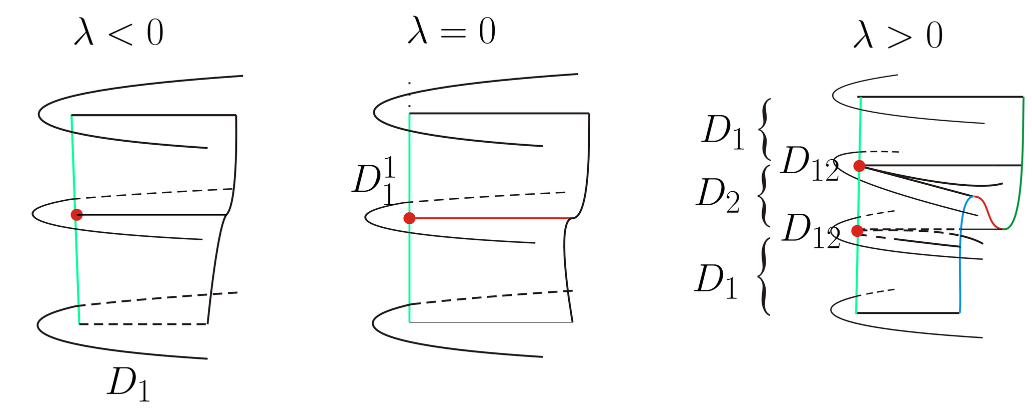


Figura 3: Unfolding of D^1_1 partially umbilic point.

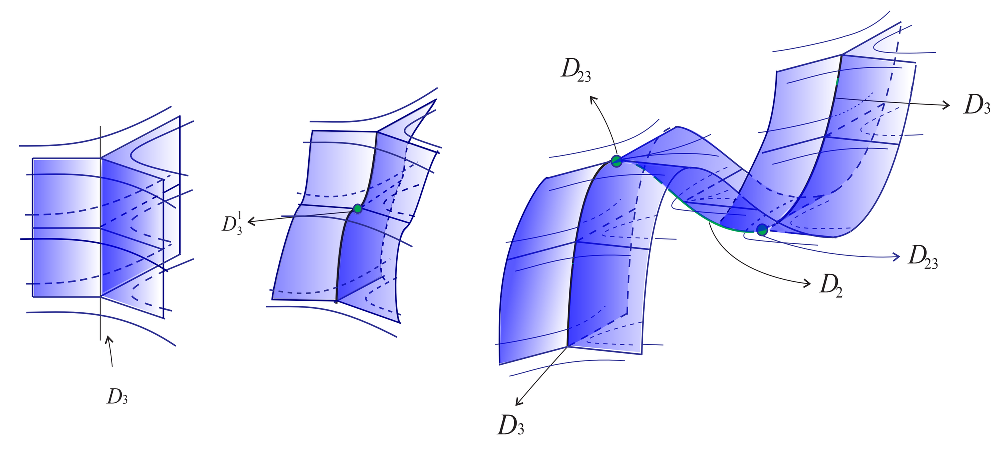


Figura 4: Unfolding of D^1_3 partially umbilic point.

Global aspect of curvature lines was studied in [2]

Referências

[1] D. Lopes, J. Sotomayor and R. Garcia, *Partially umbilic singularities of hypersurfaces of* $\mathbb{R}^4$. Bull. Sci. Math. **139** (2015), no. 4, 431-472.

[2] D. Lopes, J. Sotomayor and R. Garcia, *Umbilic Singularities and Lines of Curvature on Ellipsoids of* $\mathbb{R}^4$. Bull. of Braz. Math. Society, **45** (2014), pp. 453-483.

[3] C. Gutierrez e J. Sotomayor, *Structural Stable Configurations of lines of Principal Curvature*, Asterisque, **98-99**, (1982), pp. 185-215.

[4] R. Garcia, C. Gutierrez e J. Sotomayor, *Birfucations of Umbilical Points and Related Principal Cycles*, Jour. Dyn. and Diff. Equations, **16** (2004), pp. 321-346.