Continuity of Lyapunov exponents for non-uniformly fiber-bunched cocycles

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1 Introduction

We are interested in how the Lyapunov exponents vary as functions of the cocycle. The main theorem in [1] establishes that for cocycles that admit invariant stable and unstable holonomies, denoted by H^s and H^u respectively, we have the following result:

$$\text{If } (\hat{A}_k, H^{s,k}, H^{u,k}) \xrightarrow{C^0} (\hat{A}, H^s, H^u), \text{ then } \lambda_+(\hat{A}_k) \to \lambda_+(\hat{A}).$$

More recently, in [4], Viana and Yang were able to prove the continuity of Lyapunov exponents in the C^0 topology for irreducible cocycles when the base is non-invertible. The main theorem in [4] suggests that the hypotheses in [1] can be relaxed.

Conjecture [M. Viana]

$$\text{If } (\hat{A}_k, H^{s,k}) \xrightarrow{C^0} (\hat{A}, H^s), \text{ then } \lambda_+(\hat{A}_k) \to \lambda_+(\hat{A}).$$

We give a partial answer to this conjecture. In the following, we are going to consider $\hat{\mu}$ as an ergodic $\hat{f}$-invariant measure with local product structure such as supp $\hat{\mu} = M$. The constant $\sigma > 1$ is determined by the dynamics in the base. We say that $\hat{A}$ is non-uniformly fiber-bunched if $\lambda_+(\hat{A}) < \frac{\log \sigma}{2}$.

Theorem 1

Let $\hat{A}\colon \hat{\Sigma} \to SL(2,\mathbb{R})$ be a Lipschitz linear cocycle such that $\hat{A}$ is non-uniformly fiber-bunched and admits uniform stable holonomies. Consider a sequence $(\hat{A}_k, H^{s,k})$ such that $\hat{A}_k \to \hat{A}$ in the Lipschitz topology and $H^{s,k} \to H^s$ in the C^0 topology. Then, $\lambda_+(\hat{A}_k) \to \lambda_+(\hat{A})$.

We remark that the hypothesis of existence of this uniform stable holonomy cannot be removed. In the discontinuity example of [2], the cocycle $\hat{A}$ can be taken non-uniformly fiber-bunched. Therefore, we cannot expect continuity of the Lyapunov exponents to hold in the space of non-uniformly fiber-bunched cocycles without some extra hypotheses.

When $\hat{M}$ is the full shift, we say that $\hat{A}$ is a *locally constant linear cocycle* if it only depends on the zeroth coordinate, and it is irreducible if there is no proper subspace of $\mathbb{R}^2$ invariant under $\hat{A}(\hat{x})$ for $\hat{\mu}$-ae.

Theorem 2

Let $\hat{A}\colon \hat{\Sigma} \to SL(2,\mathbb{R})$ be a Lipschitz linear cocycle such that $\hat{A}$ is non-uniformly fiber-bunched, locally constant and irreducible. If $\hat{A}_k \to \hat{A}$ in the Lipschitz topology, then $\lambda_+(\hat{A}_k) \to \lambda_+(\hat{A})$.

2 Preliminares

Let $\hat{\Sigma}$ be a sub-shift of finite type and $\hat{f} : \hat{\Sigma} \to \hat{\Sigma}$ be the left-shift map defined by $\hat{f} = (x_{n+1})_{n\in}$ and let $\hat{A} : \hat{\Sigma} \to SL(2,)$ be a Lipchitz map, then $\hat{A}$ defines a linear cocycle $F_{\hat{A}} : \hat{\Sigma} \times \mathbb{R}^2 \to \hat{\Sigma} \times \mathbb{R}^2$ over $\hat{f}$ by:

$$F_{\hat{A}}(\hat{x}, v) = (\hat{f}(\hat{x}), \hat{A}(\hat{x})v).$$

For every $\hat{f}$-invariant probability measure $\hat{\mu}$, the Subaditive Ergodic Theorem states that

$$\lambda_+(\hat{A},\hat{x}) = \lim_{n\to\infty} \frac{1}{n} \log \|\hat{A}^n(\hat{x})\| \text{ and } \lambda_-(\hat{A},\hat{x}) = \lim_{n\to\infty} \frac{1}{n} \log \|(\hat{A}^n(\hat{x}))^{-1}\|^{-1},$$

are well defined $\hat{\mu}$-almost every $\hat{x} \in \hat{\Sigma}$. They are called extremal Lyapunov exponents of $\hat{A}$ and are $\hat{f}$-invariant, so if $\hat{\mu}$ is ergodic, they are constant $\hat{\mu}$-almost everywhere.

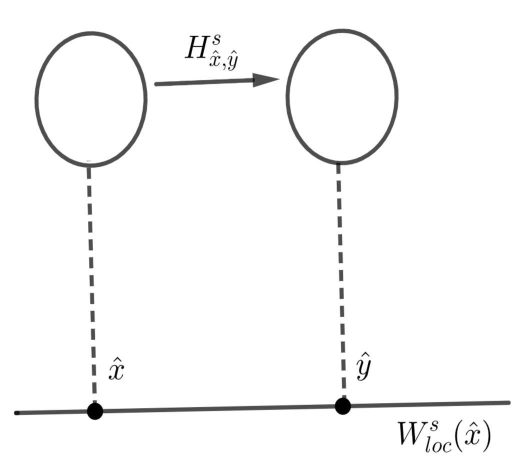

Figure 1: Definition of $H^s_{\hat{x},\hat{y}}$

3 Invariant holonomies

Definition 3.1. *An **uniform stable holonomy** for $\hat{A}$ over $\hat{f}$ is a collection of linear maps $H^s_{\hat{x},\hat{y}}$ over $\mathbb{R}^2$ or $\mathbb{P}^1$ as in Figure 1, such that*

1. Transitive.

2. Commutes with the cocycle.

3. $(\hat{x},\hat{y},\xi) \mapsto H^s_{\hat{x},\hat{y}}(\xi)$ is continuous.

Definition 3.2. *A **non-uniform stable holonomy** for $\hat{A}$ over $\hat{f}$ is a collection of linear maps $H^s_{\hat{y},\hat{z}}$ defined on a $\hat{\mu}$-full measure set M^s sastifying 1. and 2. of Definition 3.1 and*

3^*. *$(\hat{y},\hat{z}) \mapsto H^s_{\hat{y},\hat{z}}$ is measurable.*

4^*. *There exists an increasing sequence of compact subsets of M^s such that in every such subset $\left\|H^s_{\hat{y},\hat{z}} - Id\right\|$ is uniformly bounded.*

Proposition 3.3. *Every non-uniformly fiber-bunched cocycle admits non-uniform invartiant holonomies. Moreover, if*

$$\hat{A}_k \to \hat{A} \text{ then } H^{s,k} \to H^s \text{ and } H^{u,k} \to H^u.$$

We define the projectivization of $F_{\hat{A}}$ by $\mathbb{P}(F_{\hat{A}})\colon \hat{\Sigma} \times \mathbb{P}^1 \to \hat{\Sigma} \times \mathbb{P}^1$, where

$$\mathbb{P}(F_{\hat{A}})(\hat{x},[v]) = (\hat{f}(\hat{x}), [\hat{A}(\hat{x})v]).$$

If $\hat{A}$ has invariant holonomies $H^s_{\hat{x},\hat{y}}$, we define the ones for $\mathbb{P}(F_{\hat{A}})$ as $h^s_{\hat{x},\hat{y}} = \mathbb{P}(H^s_{\hat{x},\hat{y}})$.

Definition 3.4. *Let $\hat{m}$ be a $\mathbb{P}(F_{\hat{A}})$-invariant probability measure such that $\pi_* \hat{m} = \hat{\mu}$. $\hat{m}$ is a u-state if there exists a $\hat{\mu}$-full measure set M^u such that*

$$(h^u_{\hat{x},\hat{y}})_* \hat{m}_{\hat{x}} = \hat{m}_{\hat{y}} \text{ for } \hat{y} \in W^u_{loc}(\hat{x}) \text{ and } \hat{x} \in M^u.$$

If a measure is simultaneously a u-state and an s-state, we will call it su-state.

Proposition 3.5. *Let $\hat{A}$ be non-uniformly fiber-bunched admiting an uniform stable holonomy. For every su-state there exists a continuous disintegration $\{\hat{m}_{\hat{z}}\}$ that is invariant by the invariant holonomies.*

4 Characterization of discontinuity points

By the semi-continuity of $\lambda_+(\cdot)$ and $\lambda_-(\cdot)$, we have that if $\hat{A}$ is a discontinuity point of $\lambda_+(\cdot)$, then $\lambda_-(\hat{A}) < 0 < \lambda_+(\hat{A})$. Let $\mathbb{R}^2 = E^{s,\hat{A}}_{\hat{x}} \oplus E^{u,\hat{A}}_{\hat{x}}$ be the Oseledets decomposition associated to $\hat{A}$ at the point $\hat{x} \in \hat{\Sigma}$. Consider the measures projecting down to $\hat{\mu}$ in $\hat{\Sigma} \times \mathbb{P}^1$ defined by

$$\hat{m}^s = \int_{\hat{\Sigma}} \delta_{(\hat{x},\mathbb{P}(E^{s,\hat{A}}_{\hat{x}}))}\, d\hat{\mu} \quad \text{and} \quad \hat{m}^u = \int_{\hat{\Sigma}} \delta_{(\hat{x},\mathbb{P}(E^{u,\hat{A}}_{\hat{x}}))}\, d\hat{\mu}. \tag{1}$$

By construction, they are both $\mathbb{P}(F_{\hat{A}})$-invariant probability measures and

$$\lambda_-(\hat{A}) = \int_{\hat{\Sigma}\times\mathbb{P}^1} \log\|\hat{A}(\hat{x})v\|\, d\hat{m}^s \quad \text{and} \quad \lambda_+(\hat{A}) = \int_{\hat{\Sigma}\times\mathbb{P}^1} \log\|\hat{A}(\hat{x})v\|\, d\hat{m}^u.$$

Proposition 4.1. *Let $\hat{A}$ be non-uniformly fiber-bunched. If $\hat{A}$ is a discontinuity point of $\lambda_+(\cdot)$, then $\hat{m}^s$ and $\hat{m}^u$ are su-states.*

4.1 Proof of Theorem 1:

Suppose that $\lambda_+(\hat{A}_k) \nrightarrow \lambda_+(\hat{A})$, then $\lambda_-(\hat{A}) < 0 < \lambda_+(\hat{A})$.

There exist a periodic point $\hat{p}$ such that $\hat{A}^{per(p)}(\hat{p})$ is hyperbolic, denote a and r its attractor and repellent directions.

Take $\hat{m}_k$ satisfying the second equation in (1) for every k.

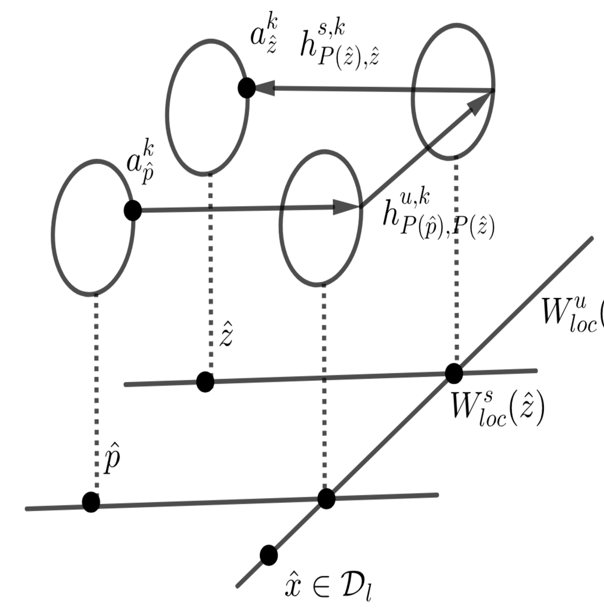

Figure 2: Definition of $a^k_{\hat{z}}$

4.1.1 Case I: $\hat{m}_k$ are su-states

The measures $\hat{m}^u$ and $\hat{m}^s$ in (1) have a disintegration satisfying that $\hat{m}^u_{\hat{z}} = \delta_{a_{\hat{z}}}$ and $\hat{m}^s_{\hat{z}} = \delta_{r_{\hat{z}}}$ such that $\hat{z} \mapsto a_{\hat{z}}$ and $\hat{z} \mapsto r_{\hat{z}}$ are continuous, then the sets $M^+ = \{(\hat{z}, a_{\hat{z}})\}_{\hat{z}\in\hat{\Sigma}}$ and $M^- = \{(\hat{z}, r_{\hat{z}})\}_{\hat{z}\in\hat{\Sigma}}$ are separated.

For k large, $\hat{A}^{n_p}_k(\hat{p})$ is also hyperbolic, let $\{a^k_{\hat{p}}, r^k_{\hat{p}}\}$ be its attractor and repellent directions. Thus $a^k_{\hat{p}} \to a$ and $r^k_{\hat{p}} \to r$.

Define $a^k_{\hat{z}}$ as in Figure 2 and $\hat{r}^k_{\hat{z}}$ analogously.

As $\hat{m}_k$ are su-states, by Proposition 3.5 we get that they have a continuous disintegration $\{\hat{m}^k_{\hat{z}}\}_{\hat{z}\in\hat{\Sigma}}$ such that supp $\hat{m}^k_{\hat{z}} \subseteq \{a^k_{\hat{z}}, \hat{r}^k_{\hat{z}}\}$.

If #supp $\hat{m}^k_{\hat{z}} = 2$, due to the separation of M^+ and M^-, we prove that both $\int \delta_{a^k_{\hat{z}}} d\hat{\mu}$ and $\int \delta_{r^k_{\hat{z}}} d\hat{\mu}$ are $\mathbb{P}(F_{\hat{A}})$-invariant measures, which contradicts the ergodicity of $\hat{m}_k$

We conclude that $\hat{m}^k_{\hat{z}}$ can only have an atom, thus $\hat{m}_k \to \hat{m}^s$ or $\hat{m}_k \to \hat{m}^u$, which contradicts that $\lambda_+(\hat{A}_k) \nrightarrow \lambda_+(\hat{A})$.

4.1.2 Case II: $\hat{m}_k$ are not su-states

In this opportunity we define

$$m_x = \int_{W^s_{loc}(x)} \hat{m}_{\hat{y}}\, d\mu^s,$$

where $\mu^s = P^s_* \hat{\mu}$, with P^s being the projection to the negative coordinates. With this definition the map $x \to m^k_x$ is continuous and $m^k_x \to m_x$ for every x. As the measures $\hat{m}_k$ are not su-states, then m_x are also non-atomic, so this allow us to use the energy argument in [1].

4.2 Proof of Theorem 2:

Firstly, we observe that because $\hat{A}$ is a locally constant cocycle, it has both uniform invariant holonomies which are in fact the identity. We suppose that, as before, $\hat{A}$ is a discontinuity point for the map $\hat{A} \mapsto \lambda_+(\hat{A})$, so we have that Equation (1) holds.

Because $\hat{m}^u$ is an su-state, $\xi : \hat{\Sigma} \to \mathbb{P}^1$ defined as $\xi(\hat{x}) =$ supp $\hat{m}^u_{\hat{x}}$ is continuous and invariant by the invariant holonomies.

Because $H^s = Id = H^u$, we have in fact that $\xi(\hat{x}) = E$ where $E \in \mathbb{P}^1$ for every $\hat{x} \in \hat{\Sigma}$.

Finally, as $\hat{m}^u$ is a $\mathbb{P}(F_{\hat{A}})$-invariant measure, we have that $(\mathbb{P}(F_{\hat{A}}))_* \hat{m}^u_{\hat{x}} = \hat{m}^u_{\hat{f}(\hat{x})}$ for $\hat{\mu}$-ae. This means that $\mathbb{P}(\hat{A})(\hat{x})E = E$ for $\hat{\mu}$-almost every point.

If the projectivization of $\hat{A}$ has a fixed point, then $\hat{A}$ has an invariant subspace, which contradicts the hypothesis of irreducibility and concludes the proof of the Theorem 2.

□

References

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