Total coloring of some Cayley graphs

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Introduction and definitions

A Cayley graph G is a graph associated to an algebraic finite group $(\mathcal{G}, *)$ and a generating set $S \subseteq \mathcal{G}$, where the vertices of the graph G correspond to the elements of the group $\mathcal{G}$ and, for elements $u, v \in \mathcal{G}$, there is the edge $(u, v) \in E(G)$ iff there is an element $s \in S$ such that $v = u*s$. The graphs considered in this work are undirected Cayley graphs and there are no loops, there is, the identity element $\iota \notin S$ and if $g \in \mathcal{G}$ then there is a unique $g' \in \mathcal{G}$ such that $g * g' = \iota$, denote g' by the inverse of g.

A k-total coloring of G is an assignment of k colors to the edges and vertices of G, such that no adjacent elements (vertices or edges) receive the same colour. The total chromatic number of G, denoted by $\chi_T(G)$, is the least k for which G has a k-total coloring. Let $\Delta(G)$ be the maximum degree of G, clearly, $\chi_T(G) \geqslant \Delta(G) + 1$ and the Total Colouring Conjecture (TCC) [1, 6] states that $\chi_T(G) \leqslant \Delta(G) + 2$. This conjecture has been verified for some classes but the general statement has remained open for more than fifty years and has not been settled even for regular graphs.

If $\chi_T(G) = \Delta(G) + 1$, then G is said to be Type 1, and if $\chi_T(G) = \Delta(G) + 2$, then G is said to be Type 2. The problem of deciding if a graph is Type 1 has been shown NP-complete [5].

The goal of this work is to present results about total coloring of some Cayley graphs, using your algebraic structure. Few Cayley graphs whose total chromatic numbers have been determined are presented as follows.

The *Cycle graphs* C_n are Cayley graphs that can be associated to an algebraic finite group $(\mathcal{Z}_n, +)$, where $\mathcal{Z}_n = \{0, 1, 2, \cdots, n-1\}$ and the generating set $S = \{1, n-1\}$.

The total chromatic number of Cycle graphs C_n are [7]:

$$\chi_T(C_n) = \begin{cases} \Delta + 1, \text{ if } n \text{ is multiple of } 3 \\ \Delta + 2, \text{ otherwise.} \end{cases}$$

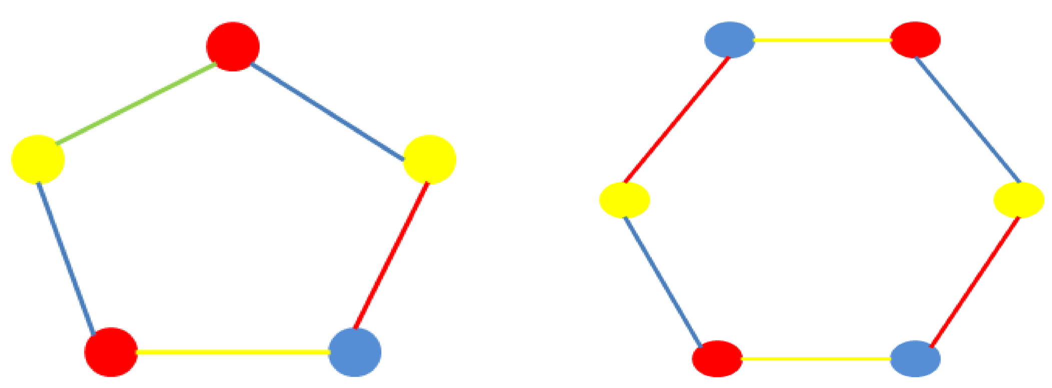

Figure 1: C_5 is Type 2 and C_6 is Type 1.

The *Complete graphs* K_n are Cayley graphs that can be associated to an algebraic finite group $(\mathcal{Z}_n, +)$, where $\mathcal{Z}_n = \{0, 1, 2, \cdots, n-1\}$ and the generating set $S = \mathcal{Z}_n - \{0\}$.

The total chromatic number of Complete graphs K_n are [7]:

$$\chi_T(K_n) = \begin{cases} \Delta + 1, \text{ if } n \text{ is odd} \\ \Delta + 2, \text{ if } n \text{ is even.} \end{cases}$$

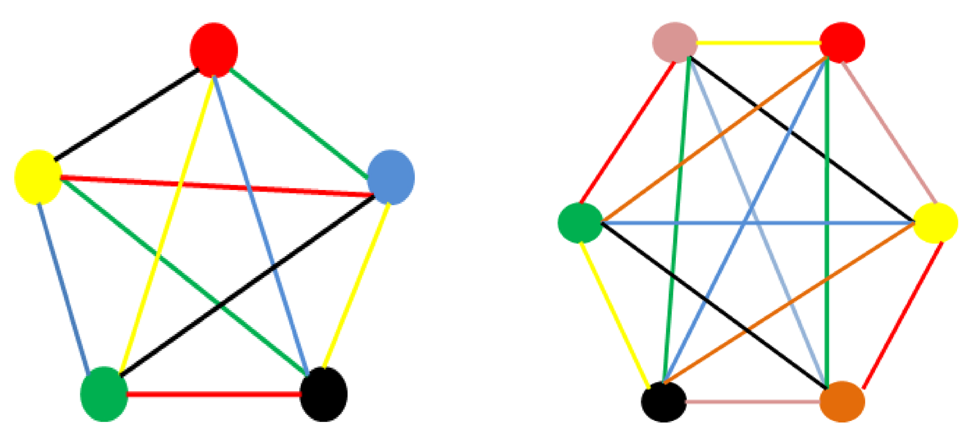

Figure 2: K_5 is Type 1 and K_6 is Type 2.

The *Power of Cycle graphs* C_n^k are Cayley graphs that can be associated to an algebraic finite group $(\mathcal{Z}_n, +)$, with the generating set $S = \{1, 2, \cdots, k, n-k, \cdots, n-1\}$.

Campos and de Mello [2] verified the TCC for power of cycles C_n^k, n even and $2 < k < \frac{n}{2}$ and also showed that one can obtain a $\Delta(G)+2$ total coloring for these graphs in polynomial time. They proved that $C_n^2, n \neq 7$, is Type 1 and C_7^2 is Type 2. They also proved that C_n^k with $n \cong 0 \mod (\Delta(C_n^k) + 1)$ are Type 1. Zorzi [8] proved that, if C_n^k is not a complete graph, then for $k = 3$ or $k = 4$, C_n^k is Type 2 if n is odd and $k > \frac{n}{3} - 1$ and Type 1 otherwise. He also proved that, for $k = 5$ or $k = 7$, C_n^k is Type 1 if $n \geqslant 4k^2 + 2k$. Geetha et al. [3] verified TCC for the complement of power of cycles $\bar{C}_n^k$.

Note that the complete graph $K_5 = C_5^2$ is Type 1.

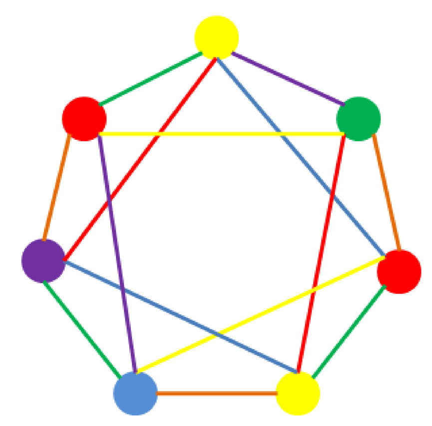

Figure 3: C_7^2 is Type 2

The *Circulant graphs* $C_n(d_1, d_2, \cdots, d_\ell)$ are Cayley graphs that can be associated to an algebraic finite group $(\mathcal{Z}_n, +)$, where $\mathcal{Z}_n = \{0, 1, 2, \cdots, n-1\}$, with a generating set $S = \{1, n-1\} \cup D$, where D is the set of d_i such that $x = (y \pm d_i) \mod n$ and x, y are vertices of the graph.

Khennoufa and Togni [4] studied total colorings of circulant graphs and proved that every 4-regular circulant graphs, for any positive integer p, $C_{5p}(1, k)$ are Type 1 for $k < \frac{5p}{2}$ with $k \equiv 2 \mod 5$ or $k \equiv 3 \mod 5$ and $C_{6p}(1, k)$ are Type 1 for $p \geqslant 3$ and $k < 3p$ with $k \equiv 1 \mod 3$ or $k \equiv 2 \mod 3$.

Note that the complete graph $K_5 = C_5(1, 2)$ is Type 1.

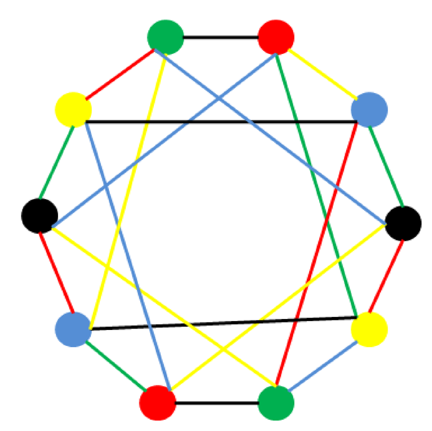

Figure 4: $C_{10}(1, 3)$ is Type 1

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