Serrin type solvability criteria for Dirichlet problems for prescribed mean curvature equations in manifolds

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Abstract

For a complete Riemannian manifold M of dimension n, we find necessary and sufficient conditions for the existence of vertical graphs in $M\times\mathbb{R}$ with prescribed mean curvature having given boundary values. Our results generalize classical results of Jenkins-Serrin and Serrin in the Euclidean ambient space, as well as Spruck's results in the $M\times\mathbb{R}$ setting.

Introduction

Given a smooth bounded domain Ω in a complete Riemannian manifold M, we study if for a given smooth function φ and a prescribed smooth function $H = H(x,z)$ non-decreasing in the variable z, there exists or not a smooth function u satisfying

$$\begin{cases} \operatorname{div}\left(\dfrac{\nabla u}{W}\right) = nH(x,u) \text{ in } \Omega, \\ u = \varphi \text{ in } \partial\Omega, \end{cases} \tag{P}$$

where $W = \sqrt{1+\|\nabla u(x)\|^2}$ and the quantities involved are calculated with respect to the metric of M.

Objectives

1. Show that the *strong Serrin conditon*
$$(n-1)\mathcal{H}_{\partial\Omega}(y) \geq n \sup_{z\in\mathbb{R}} |H(y,z)| \ \forall\, y \in \partial\Omega, \tag{1}$$
is necessary for the solvability of problem (P).
2. Study under which conditions on the function H (1) is also sufficient.

Non-existence results

Theorem 1. *Let M be a Cartan-Hadamard manifold and $\Omega \subset M$ a $\mathcal{C}^2$ bounded domain. Let $H \in \mathcal{C}^0(\overline{\Omega}\times\mathbb{R})$ be a function either non-negative or non-positive and non-decreasing in the variable z. If the strong Serrin condition* (1) *fails, then there exists $\varphi \in \mathcal{C}^\infty(\overline{\Omega})$ such that there is no $u \in \mathcal{C}^0(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$ satisfying problem* (P).

Theorem 2. *Let M be a simply connected and compact manifold with $0 < \frac{1}{4}K_0 < K \leq K_0$. Let $\Omega \subset M$ be a $\mathcal{C}^2$ domain with $\operatorname{diam}(\Omega) < \frac{\pi}{2\sqrt{K_0}}$. Let $H \in \mathcal{C}^0(\overline{\Omega}\times\mathbb{R})$ be a function either non-negative or non-positive and non-decreasing in the variable z. If the strong Serrin condition* (1) *fails, then there exists $\varphi \in \mathcal{C}^\infty(\overline{\Omega})$ such that there is no $u \in \mathcal{C}^0(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$ satisfying problem* (P).

Combining theorem 2 with the existence result from Aiolfi-Ripoll-Soret [1, Th. 1] for the minimal case we observed that the Jenkins-Serrin's theorem [2, Th. 1] also holds in every Hadamard manifold:

Theorem 3 (Sharp Jenkins-Serrin-type solvability criterion). *Let M be a Cartan-Hadamard manifold and $\Omega \subset M$ a $\mathcal{C}^{2,\alpha}$ bounded domain. Then for $H = 0$ the Dirichlet problem* (P) *has a unique solution for arbitrary continuous boundary data if, and only if, Ω is mean convex.*

Combining theorem 1 with a result of Spruck [4, Th. 1.4] we derived:

Theorem 4 (Sharp Serrin-type solvability criterion 1). *Let M be a simply connected and compact manifold with $0 < \frac{1}{4}K_0 < K \leq K_0$. Let $\Omega \subset M$ be a $\mathcal{C}^{2,\alpha}$ domain with $\operatorname{diam}(\Omega) < \frac{\pi}{2\sqrt{K_0}}$. Then for every constant H the Dirichlet problem* (P) *in Ω has a unique solution for arbitrary continuous boundary data if, and only if, $(n-1)\mathcal{H}_{\partial\Omega} \geq n\,|H|$.*

Existence results

In order to obtain sharp Serrin-type solvability criteria in Cartan Hadamard manifolds, we have established the following existence result:

Theorem 5. *Let $\Omega \subset \mathbb{H}^n$ be a $\mathcal{C}^{2,\alpha}$ bounded domain and $\varphi \in \mathcal{C}^{2,\alpha}(\overline{\Omega})$. Let $H \in \mathcal{C}^{1,\alpha}(\overline{\Omega}\times\mathbb{R})$ satisfying $\partial_z H \geq 0$ and $\sup_{\Omega\times\mathbb{R}} |H| \leq \frac{n-1}{n}$. If the strong Serrin condition* (1) *holds, then for every $\varphi \in \mathcal{C}^{2,\alpha}(\overline{\Omega})$ there exists a unique solution $u \in \mathcal{C}^{2,\alpha}(\overline{\Omega})$ of the Dirichlet problem* (P).

Results previous to theorem 5

1. Spruck [4, Th. 5.4]: $H \in \left[0, \frac{n-1}{n}\right]$ and $(n-1)\mathcal{H}_{\partial\Omega} > n\,|H|$.

By combining our theorems 1 and 5 with a theorem from Spruck [4, Th. 1.4] in the case where $H \geq \frac{n-1}{n}$ we derived:

Theorem 6 (Sharp Serrin-type solvability criterion in hyperbolic space). *Let $\Omega \subset \mathbb{H}^n$ be a $\mathcal{C}^{2,\alpha}$ bounded domain. Then for every constant H the Dirichlet problem* (P) *has a unique solution for arbitrary continuous boundary data if, and only if, $(n-1)\mathcal{H}_{\partial\Omega}(y) \geq n\,|H|$.*

The main existence result is the following:

Theorem 7. *Let $\Omega \subset M$ be a $\mathcal{C}^{2,\alpha}$ bounded domain. Let $H \in \mathcal{C}^{1,\alpha}(\overline{\Omega}\times\mathbb{R})$ satisfying $\partial_z H \geq 0$,*

$$\operatorname{Ricc}_x \geq n \sup_{z\in\mathbb{R}} \|\nabla_x H(x,z)\| - \frac{n^2}{n-1} \inf_{z\in\mathbb{R}} (H(x,z))^2 \ \forall\, x \in \Omega$$

and the strong Serrin condition (1)*. Then for every $\varphi \in \mathcal{C}^{2,\alpha}(\overline{\Omega})$ there exists a unique solution $u \in \mathcal{C}^{2,\alpha}(\overline{\Omega})$ of problem* (P).

Results previous to theorem 7

1. Serrin [3, Th. p. 484]: $M = \mathbb{R}^n$ and $H = H(x)$.
2. Spruck [4, Th. 1.4]: $H \in \mathbb{R}^+$.

Combining theorem 7 with theorems 1 and 2 also obtain sharp solvability criteria for the Dirichlet problem (P).

Referências

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