Second variation of the Hausdorff measure of non-horizontal hypersurfaces in sub-Riemannian stratified Lie groups

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Abstract

We determine stability conditions for a non-horizontal hypersurfaces of a sub-Riemannian stratified Lie group. We calculate the second variation of the measure for a non-horizontal hypersurfaces and discuss some examples of stable hypersurfaces in the Heisenberg group.

Introduction

A stratified Lie group $\mathbb{G}$ is an n-dimensional connected, simply connected nilpotent Lie group whose Lie algebra $\mathfrak{g}$ decomposes as $\mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}^2 \oplus \cdots \oplus \mathfrak{g}^r$ and satisfies the condition $[\mathfrak{g}^1, \mathfrak{g}^j] = \mathfrak{g}^{j+1}, j = 1, \ldots, r-1, [\mathfrak{g}^j, \mathfrak{g}^r] = 0, j = 1, \ldots, r$. If $\langle \cdot, \cdot \rangle$ is a scalar product on $\mathfrak{g}^1$, we can extend it to $\mathfrak{g}$ by induction, see [2]. We also consider the distribution $D \subset T\mathbb{G}$ generated by $\mathfrak{g}^1$ and the scalar product in D generated by $\langle \cdot, \cdot \rangle$. This way, $(\mathbb{G}, D, \langle \cdot, \cdot \rangle)$ becomes a sub-Riemannian manifold, also called the Carnot group.

We denote by $\overline{\nabla}$ the covariant derivative defined by $\overline{\nabla} X = 0$ for all $X \in \mathfrak{g}$. It has intrinsic torsion which is essentially the negative of the Lie bracket in $\mathbb{G}$ and the zero curvature tensor. So, this covariant derivative permits us to establish an interesting parallel between the invariants of submanifolds in $\mathbb{R}^n$ and the invariants of submanifolds in $\mathbb{G}$.

In [2] it was to used the measure proposed by Magnani and Vittone [3] for non-horizontal submanifolds in stratified Lie groups, to calculate the first variation of the measure for these submanifolds and determine the necessary conditions for a non-horizontal submanifold to be of minimal measure. Our purpose here is to study the stability conditions for minimal non-horizontal hypersurfaces in stratified Lie groups.

We say that $M \subset \mathbb{G}$ a non-horizontal hypersurface if $TM + D = T\mathbb{G}$ (M is transverse to D). We use an adapted basis to M which is a crucial step to obtain our results. Let $f_1, \ldots, f_n$ be the adapted basis in the $T\mathbb{G}$ such that f_1 is orthogonal to $TM \cap D$ and $f_2, \ldots, f_{d_1}$ is an orthonormal basis of $TM \cap D$ in D. We complete $f_2, \ldots, f_{d_1}$ to a basis $f_2, \ldots, f_n$ of TM taking $f_j = e_j - A^1_j f_1$, for $j = d_1 + 1, \ldots, n$. It was proved in [2] that the density of the measure μ on M it has the simple expression $d\mu = \mathrm{f}^{p+1} \wedge \cdots \wedge \mathrm{f}^n$. We can easily be seen that $d\mu$ is parallel respect to projected connection on TM. Consequently, we introduce the sub-Laplacian operator by $\mathcal{L}\phi = \Delta\phi + \tau\phi$, where Δ is the horizontal Laplacian operator on $TM \cap D$ and $\tau\phi = \sum_{j=d_1+1}^{n} \overline{T}^j(\nabla\phi, f_j)$, where ϕ is smooth function on M. It is easily proved that $\int_M \phi\mathcal{L}\phi = -\int_M ||\nabla\phi||^2$.

Main result

Let M be an oriented manifold of dimension $n-1$, $\mathbb{G}$ a stratified Lie group of dimension n and $i : M \to \mathbb{G}$ an immersion of M in $\mathbb{G}$ as a non-horizontal hypersurface. Let $F : (-\epsilon, \epsilon) \times M \to \mathbb{G}$ be a variation of i through immersions, with variation vector field W.

In [2], it was proved that M is minimal if $H_\xi = 0$, where $H_\xi = -trA_\xi$, $\xi \in TM^\perp$. Consider $W^\perp = \psi f_1$ (orthogonal component of variation vector field W), where $\psi : \mathbb{G} \to \mathbb{R}$ is a smooth function and $\psi|_{\partial M} = 0$. If $V(u)$ denotes the element volume on M, then

$$\ddot{V}(0) = \int_M \|\nabla\psi\|^2 + \psi^2 \left(-||A_{f_1}||^2 + tr(A_{f_1} \circ (f_1 \lrcorner \overline{T}^\top)) + \langle \mathfrak{B}(f_1), f_1 \rangle \right)$$

where $\nabla\psi$ denotes the vector field gradient on $TM \cap D$ and $\langle \mathfrak{B}(f_1), f_1 \rangle = \sum_{i=2}^{d_1} \langle f_1, S(f_i, \overline{T}^\top(f_1, f_i)) \rangle$.

We can say that M is stable if $\ddot{V}(0) \geq 0$.

Applications

Application 1. *Let M be a minimal non-horizontal hypersurface of $\mathbb{G}$ and let $\zeta \in \mathfrak{g}^1 \subset D$ such that $\zeta^\perp \neq 0$ on M. Then,*

$$\ddot{V}(0) = \int_M \left(\left\| h\nabla\left(\frac{\psi}{h}\right) \right\|^2 - \frac{\psi^2}{h}\mathcal{S} \right), \forall \psi \in C_0^\infty(M),$$

where $h = \langle \zeta, f_1 \rangle$. In particular, M is stable if $\frac{1}{h}\mathcal{S} \leq 0$.

Application 2. *Let the assumptions of Application 1. If M is vertical, then M is stable.*

Application 3. *Let $M \subset \mathbb{H}^2$ be a minimal non-horizontal hypersurface. Then,*

$$\ddot{V}(0) = \int_M \left(\|\nabla\psi\|^2 + 2\psi^2 \left(-\mathcal{K} + \mathrm{dA}^1_5(\mathrm{f}_3) - (\mathrm{A}^1_5)^2 \right) \right),$$

$\forall \psi \in C_c^\infty(M)$, where $\mathcal{K}$ is horizontal scalar curvature of M.
In particular, M is stable if

$$-\mathcal{K} + \mathrm{dA}^1_5(\mathrm{f}_3) - (\mathrm{A}^1_5)^2 \geq 0\,.$$

Let $M = \left\{ (x_1, x_2, x_3, x_4, x_5) \in \mathbb{H}^2; x_5 = \dfrac{x_1^2 - x_3^2 + x_2^2 - x_4^2}{4} \right\}$ be the hyperbolic paraboloid. Then, $M - \{p\}$ is stable, where $p = (x_1, x_2, -x_1, -x_2, 0)$.

Application 4. *Let $S \subset \mathbb{H}^1$ be a minimal non-horizontal surface. Then,*

$$\ddot{V}(0) = \int_S \left(\|\nabla\psi\|^2 + \psi^2 \left(2\mathrm{dA}^1_3(\mathrm{f}_2) - (\mathrm{A}^1_3)^2 \right) \right), \forall \psi \in C_0^\infty(S).$$

In particular, S is stable if $2\mathrm{dA}_3(\mathrm{f}_2) - (\mathrm{A}^1_3)^2 \geq 0$.

Let us consider a smooth parametrization $\phi : \mathcal{D} \to S$ given by $\phi(x, y) = (x, y, v(x, y))$, where $v : \mathcal{D} \to \mathbb{R}$ is a smooth function and $\mathcal{D} \subset \mathbb{R}^2$ is an open domain. Using application 4, we prove that $v(x, y) = -\dfrac{xy}{2} + \nu(y)$ and $v(x, y) = \dfrac{xy}{2} + \nu(x)$, where $\nu : \mathbb{R} \to \mathbb{R}$ is smooth function, are stables surfaces in $\mathbb{H}^1$. These are examples of graph invariants under left translations by one parameter subgroups of the Heisenberg group, see [1].

References

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