Enneper representation of minimal surfaces in the three-dimensional Lorentz-Minkowski space.

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Introduction

The Weierstrass representation formula for minimal surfaces in $\mathbb{R}^3$ is a powerful tool to construct examples and to prove general properties of such surfaces, since it gives a parametrization of minimal surfaces by holomorphic data. In [5] the authors described a general Weierstrass representation formula for simply connected immersed minimal surfaces in an arbitrary Riemannian manifold. The partial differential equations involved are, in general, too complicated to find explicit solutions. However, for particular ambient 3-manifolds, such as the Heisenberg group, the hyperbolic space and the product of the hyperbolic plane with $\mathbb{R}$, the equations become simpler and the formula can be used to construct examples of conformal minimal immersions ([5]).

In [1], Andrade introduces a new method to obtain minimal surfaces in the Euclidean 3-space which is equivalent to the classical Weierstrass representation and, also, he proves that any immersed minimal surfaces in $\mathbb{R}^3$ can be obtained using it.

Theorem 1. *Let $h : \Omega \to \mathbb{R}$ be a harmonic function in the simply connected domain $\Omega \subset \mathbb{C}$ and $L, P : \Omega \to \mathbb{C}$ two holomorphic functions such that the following conditions are satisfied:*

$$(h_z)^2 = L_z\, P_z \quad \text{and} \quad |L_z| + |P_z| \neq 0. \tag{1}$$

Then, the map $\psi : \Omega \to \mathbb{C} \times \mathbb{R}$, given by $\psi(z) = (L(z) - \overline{P(z)}, h(z))$, defines a conformal minimal immersion into $\mathbb{R}^3$.

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Some extensions of the Enneper-type representation in others ambient spaces have been given in [2] and [6].

The aim of this poster is to illustrate an Enneper-type representation for timelike minimal surfaces in the Lorentz-Minkowski space $\mathbb{L}^3$, i.e. the affine three space $\mathbb{R}^3$ endowed with the Lorentzian metric

$$g = dx_1^2 + dx_2^2 - dx_3^2.$$

1 Results

For timelike immersions we obtain the following results:

Theorem 2. *Let $h : \Omega \to \mathbb{R}$ be a harmonic function in the simply connected domain $\Omega \subset \mathbb{L}$ and $L, P : \Omega \to \mathbb{L}$ two $\mathbb{L}$-differentiable functions such that the following conditions are satisfied:*

$$(h_z)^2 = L_z\, P_z \tag{2}$$

and

$$2\, h_z\, \overline{h_z} + L_z\, \overline{L_z} + P_z\, \overline{P_z} \neq 0. \tag{3}$$

Then, the map $\psi : \Omega \to \mathbb{R} \times \mathbb{L}$, given by $\psi(z) = (h(z), L(z) - \overline{P(z)})$, defines a conformal timelike minimal immersion into $\mathbb{L}^3$.

We will call $\psi = (h, L - \overline{P})$ an *Enneper timelike immersion* associated to h and $\mathcal{D}^{\mathbb{L}}_{\psi} = (L_z, P_z, h_z)$ the *Enneper paracomplex data* of ψ.

Theorem 3. *Let $\mathcal{M}^2$ a timelike minimal surface in $\mathbb{L}^3$, given by the immersion $\psi : \Omega \to \mathbb{L}^3$, where $\Omega \subset \mathbb{L}$ is a simply connected domain. Then, there exists a harmonic function $h : \Omega \subset \mathbb{L} \to \mathbb{R}$ such that the immersed minimal surface $\mathcal{M}$ is an Enneper graph of h.*

Now, we will use Theorem 3 to provide a description of the timelike catenoids given in [4] in terms of their paracomplex Enneper data:

Catenoid	L_z	P_z	h_z
Elliptic:			
$x_1^2 + x_2^2 = (\cos x_3)^2$	$\dfrac{\tau\,(1+\cos z)}{2}$,	$\dfrac{\tau\,(1-\cos z)}{2}$	$-\dfrac{\sin z}{2}$
Hyp. of 1st kind:			
$x_2^2 - x_3^2 = (\cosh x_1)^2$	$\dfrac{\sinh z - \cosh z}{2}$	$-\dfrac{\sinh z + \cosh z}{2}$	$\dfrac{1}{2}$
Hyp. of 2nd kind:			
$x_3^2 - x_1^2 = (\sinh x_2)^2$	$\dfrac{\tau \cosh z + 1}{2}$	$\dfrac{\tau \cosh z - 1}{2}$	$\dfrac{\tau \sinh z}{2}$
Parabolic:			
$12(x_3^2 - x_1^2 - x_2^2) = (x_1 - x_3)^4$	$-\dfrac{\tau(z+1)^2}{2}$	$-\dfrac{\tau(z-1)^2}{2}$	$\dfrac{1-z^2}{2}$

2 Construction of new minimal surfaces in $\mathbb{L}^3$

We start observing that if $\mathcal{D}^{\mathbb{L}}_{\psi} = (L_z, P_z, h_z)$ are the Enneper data of a given timelike minimal immersion ψ in $\mathbb{L}^3$ (defined in the simply connected domain $\Omega \subset \mathbb{L}$) and $f : \Omega \to \mathbb{L}$ is a $\mathbb{L}$-differentiable function so that $f(z)\overline{f(z)} \neq 0$ in Ω, then

$$f\, \mathcal{D}^{\mathbb{L}}_{\psi} = (f\, L_z, f\, P_z, f\, h_z)$$

are Enneper data of a new timelike)minimal surface in $\mathbb{L}^3$. We note that this surface is the Enneper graph of the harmonic function $h_1 : \Omega \subset \mathbb{L} \to \mathbb{R}$ defined by:

$$h_1(z) = h_1(z_0) + 2\,\mathcal{R}e \int_{z_0}^{z} f(z)\, h_z(z)\, dz.$$

Also, the Enneper timelike minimal immersion associated to h_1 is given by:

$$\psi_1 = (h_1, L_1 - \overline{P_1}) \tag{4}$$

where

$$L_1(z) := \int_{z_0}^{z} f(z)\, L_z(z)\, dz, \qquad P_1(z) := \int_{z_0}^{z} f(z)\, P_z(z)\, dz, \tag{5}$$

are well-defined $\mathbb{L}$-differentiable functions in Ω.

Example 1. *Next we are going to produce a special family of Lorentzian minimal surfaces in $\mathbb{L}^3$. We consider $n \in \mathbb{Z}$, $n > 1$, and the family of paracomplex Enneper data given by:*

$$\mathcal{D}^{\mathbb{L}}_{\psi_n} = \Big(\sin z\,(1 + \tau \sin(nz)), \sin z\,(1 - \tau \sin(nz)), \sin z \cos(nz)\Big).$$

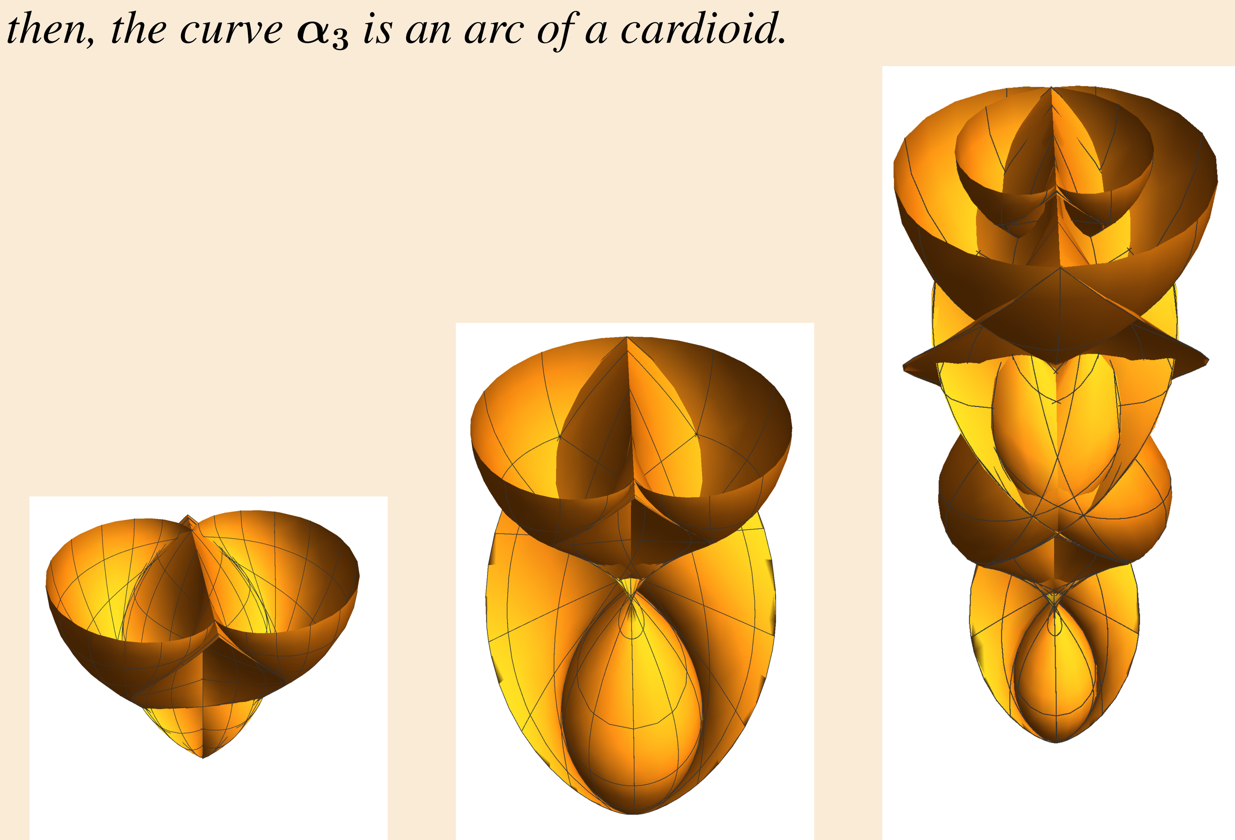
In this case, using (4) and (5), we obtain the following family of timelike minimal surfaces $\psi_n(u,v) =$

$$\Big(\frac{\cos[(n-1)u]\, \cos[(n-1)v]}{n-1} - \frac{\cos[(n+1)u]\, \cos[(n+1)v]}{n+1}, \frac{\cos[(n-1)u]\, \sin[(n-1)v]}{n-1} - \frac{\cos[(n+1)u]\, \sin[(n+1)v]}{n+1}, 2 \sin u \sin v\Big),$$

where $u \in (-\pi/4n, \pi/4n)$ and $v \in (\pi/4n, 3\pi/4n)$. Given $n \in \mathbb{Z}$, $n > 1$, we have that $\psi_n(u,v)$ is the only minimal immersion into $\mathbb{L}^3$ containing the spacelike curve $\alpha_n(v) := \psi_n(0,v)$, as a planar pregeodesic. If we consider the change of parameter $t = (n-1)\, v$, we have that

$$\alpha_n(t) = \Big(\frac{\cos t}{n-1} - \frac{\cos\left(\frac{n+1}{n-1}\, t\right)}{n+1}, \frac{\sin t}{n-1} - \frac{\sin\left(\frac{n+1}{n-1}\, t\right)}{n+1}, 0\Big),$$

that is an epycicloid traced by a point on a circle of radius $r = 1/(n+1)$ which rolls externally on a circle of radius $R = 2/(n^2-1)$. We observe that if $n = 2$, then $R = 2r$, therefore the curve α_2 is an arc of a nephroid. Also, if $n = 3$ we have that $R = r$ and, then, the curve α_3 is an arc of a cardioid.



Timelike minimal surfaces in $\mathbb{L}^3$ containing a nephroid, a cardioid and the epycicloid for $n = 5$ (respectively) as a pregeodesic.

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