# Moving Planes Method


## Fiorella Rendón

PUC-Rio

fiorellareg@mat.puc-rio.br



**Abstract**

As a first beautiful illustration of the power of the Maximum Principle, we will prove the following symmetry theorem, which plays an important role in the study of nonlinear elliptic and parabolic PDE.


**Introduction**

The method of moving planes is used in proving symmetry, in fact, $x_1$ direction for solutions of nonlinear elliptic equation $F(x, u, Du, D^2u) = 0$ in a bounded domain $\Omega$ in $\mathrm{R}^n$ which is convex in the $x_1$ direction.

The essential ingredient in their use is the Maximum principle. In this method one point is the reflection of the other in a hyperplane $\{x_1 = \lambda\}$, and then, the plane is moved up to a critical position.

We choose a very simple example to illustrate such a method. The following result was first proved by Gidas, Ni and Nirenberg.

**Theorem.1.**

Suppose $u \in C(B_1) \cap C^2(B_1)$ is a positive solution of

$$\triangle u + f(u) = 0 \text{ in } B_1$$
$$u = 0 \text{ on } \partial B_1$$

where $f$ is locally Lipschitz in R. Then $u$ is radially symmetric in $B_1$ and $\frac{\partial u}{\partial r}(x) < 0$ for $x \neq 0$.

The original proof requires that solutions be $C^2$ up to the boundary. Here we give a method which does not depend on the smoothness of domains nor the smoothness of solutions up to the boundary.

**Proof**

Before stating the result let us first fix some notations. We will be moving planes in the $x_1$-direction but of course also any other direction will do.

Some sets that will be used are:

The moving plane: $T_\lambda = \{x \in R^n; x_1 = \lambda\}$;

The subdomain: $\Sigma_\lambda = \{x \in \Omega; x_1 > \lambda\}$;

The reflected point: $x_\lambda = (2\lambda - x_1, x_2, \cdots, x_n)$;

the reflected subdomain: $\Sigma'_\lambda = \{x_\lambda; x \in \Sigma_\lambda\}$ ;

The starting value for $\lambda$: $\lambda_0 = \inf\{x_1; x \in \Omega\}$;

The maximal value for $\lambda$:

$\lambda_* = \inf\{\lambda; w_\mu < 0\Sigma_\mu \quad \text{for all} \quad \mu > \lambda\}$;

We will prove $u(x_1, y) < u(x_1^*, y)$ for any $x_1 > 0$ and $x_1* < x_1$ with $x_1 * + x_1 > 0$.

Then by letting $x_1* \to -x_1$, we get $u(x_1, y) \leq u(-x_1, y)$ for any $x_1$.

Then by changing the direction $x_1 \to -x_1$, we get the symmetry.

In $\Sigma_\lambda$ we define

$$w_\lambda(x) = u(x) - u(x_\lambda) \quad \text{for} \quad x \in \Sigma_\lambda$$

Since $\triangle$ is $x_1$-invariant, $u(x_\lambda)$ satisfies the same equation as u in $\Sigma_\lambda$:

$$\triangle u(x_\lambda) + f(u(x_\lambda)) = 0 \quad \text{in} \quad \Sigma_\lambda$$
$$\triangle w(x_\lambda) + f(u(x_\lambda)) - f(u) = 0 \Leftrightarrow \triangle w(x_\lambda) + c_\lambda(x).w_\lambda = 0$$

where

$$c_\lambda(x) = \frac{f(u(x_\lambda)) - f(u(x))}{u(x_\lambda) - u(x)} \quad \text{if} \quad u(x_\lambda) \neq u(x)$$
$$0 \quad \text{if} \quad u(x_\lambda) = u(x)$$

The hypothesis that $f$ is locally Lipschitz guarantees that $c_\lambda(x)$ is a bounded function, i .e.

$|c_\lambda(x)| \leq Lip(f)$.

Then we have by The mean value theorem

$$\triangle w_\lambda + c_\lambda(x) = 0 \quad \text{in} \quad \Sigma_\lambda$$
$$w_\lambda \leq 0 \quad \text{and} \quad w_\lambda \neq 0 \quad \text{on} \quad \partial\Sigma_\lambda$$

Moreover $\lambda < \lambda^*$ we have $u(x_\lambda) \leq u(x)$ on $\partial\Omega \cap \partial\Sigma_\lambda$ and $u(x_\lambda) = u(x)$ on $\Omega \cap \partial\Sigma_\lambda$.

Hence $w_\lambda \leq 0 \quad \text{on} \quad \partial\Sigma_\lambda$

The two basic ingredients in the proof are the maximum principle for small domains and the strong maximum principle for nonpositive functions.

Let $\delta = \delta(n, K) > 0$ be the constant from the maximum principle in small domains.

We will divide the proof in two steps:

1. The moving process can start, that is, $\exists\epsilon_1$ such that $w_\lambda < 0$ on $\Sigma_\lambda$ if $\lambda \in (a - \epsilon_1, a)$.
   Let $\epsilon_1 > 0$ be small enough such that $|\Sigma_\lambda| < \delta$ if $\lambda \in (a - \epsilon_1, a)$.
   We know that $w_\lambda = 0$ on $T_\lambda \cap \partial\Sigma_\lambda$ e $w_\lambda < 0$ on $\partial\Omega \cap \partial\Sigma_\lambda$, so we can use (1) and (2) with $z = w_\lambda$

$$\triangle w_\lambda + c_\lambda(x)w_\lambda = 0 \quad \text{in}\Sigma_\lambda \qquad (1)$$
$$w_\lambda \leq 0 \quad \text{on} \quad \partial\Sigma_\lambda \qquad (2)$$

$$\Rightarrow w_\lambda \leq 0 \quad \text{in} \quad \Sigma_\lambda$$

   By the strong M.P. $w_\lambda \equiv 0$ or $w_\lambda < 0$ on $\Sigma_\lambda$.
   But $w_\lambda \equiv 0$ is not possible, because if $w_\lambda \equiv 0$ for some $\lambda \in (a - \epsilon_1, a)$ then $u(x) = 0$ for $x \in \partial(\Sigma'_\lambda \cup \Sigma_\lambda)$ and moreover, since part of this boundary, $\Omega \cap \partial(\Sigma'_\lambda \cup \Sigma_\lambda)$ lies inside $\Omega$ it contradicts $u > 0$ in $\Omega$.
   Thus $w_\lambda < 0$ in $\Sigma_\lambda$.
   This implies in particular that $w_\lambda$ assumes along $\partial\Sigma_\lambda \cap \Omega$ its maximum in $\Sigma_\lambda$.
   We can start moving the hyperplaness.
2. If we have rechead position $\lambda > 0$, we can move a bit to the left, our objective is $\lambda_* = 0$.

The conclusion that $\frac{\partial}{\partial x_1}u(x) < 0$ for $x \in \Sigma^*_\lambda$ follows from the fact that $w_\lambda(x) < 0$ on $\Sigma_\lambda$ for all $\lambda \in (\lambda_0, \lambda^*)$ and by Hopfs boundary point Lemma $\frac{\partial}{\partial x_1}u(x) = 1/2\frac{\partial}{\partial x_1}w_\lambda(x) < 0$ on $T_\lambda \cap \Omega$

**Proof of Theorem**

From the previous theorem we find that $x_1\frac{\partial}{\partial x_1}u(x) < 0$ for $x \in B_R(0)$ with $x_1 \neq 0$. Hence $\frac{\partial}{\partial x_1}u(x) = 0$ for $x_1 = 0$. Since $\triangle$ is radially invariant this holds for every direction and we find $\frac{\partial}{\partial \tau}u(x) = 0$ in $B_R(0)$ for any tangential direction. In other words, u is radially symmetric. Since $\frac{\partial}{\partial|x|}u(x) = \frac{\partial}{\partial \tau}u(r, 0, \cdots, 0)$ for $0 < r = |x| < R$ the second claim follows from $\frac{\partial}{\partial x_1}u(x) < 0$ for $x \in B_R(0)$ with $x_1 > 0$

**Observation 1.** This result is also valid if we replace the Laplacian by an operator invariant with respect to reflections in the $x_1$ direction.

**Observation 2.** Applying Lemma.1. in all directions we obtain Theorem 1.

## Referências


[1] Q. HAN; F.LIN - Elliptic Partial Differential Equation. *AMS*, New York,2000.

[2] H. BERESTYCKI; L. NIRENBERG - On the method of moving planes and the sliding method*Bol. Soc. Bras. Mat.*, Vol. 22 No. 1, 1-37 0 1991, Sociedade Brasileira de MatemAtica, 19791.

[3] B. SIRAKOV,-Modern theory of nonlinear elliptic PDE *Notes to a course given at the 30th Colóquio Brasileiro de Matem atica at IMPA.* 2015