Geometric regularity theory for nonlinear PDEs

Edgard A. Pimentel

Department of Mathematics

Pontifical Catholic University of Rio de Janeiro, PUC-Rio

pimentel@puc-rio.br

Abstract

In this poster we put forward recent developments in regularity theory for some classes of nonlinear PDEs. Our arguments relate a problem of interest to another one, for which a richer theory is available. It operates in two distinct layers; first compactness builds upon suitable notions of stability to produce approximation results. Then, a scaling argument localizes the analysis to establish (in some cases, sharp) regularity results. The toy-models we cover include fully nonlinear PDEs, the Isaacs equation, double-divergence problems and degenerate/singular equations.

Key-words: Regularity theory; Regularity transmission by approximation methods; Estimates in Hölder and Sobolev spaces.

Regularity theory for nonlinear operators

Regularity theory is about the properties of a function which are implied by the mere fact that it **solves** a given partial differential equation, in some appropriate weak sense.

In the linear case, a celebrated result is the so-called **Schauder's regularity theory**. It says that weak solutions to

$$\Delta u = f \quad \text{in} \quad B_1$$

are of class $\mathcal{C}^{2\alpha}_{loc}(B_1)$, provided $f \in \mathcal{C}^{\alpha}_{loc}(B_1)$.

A further example os regularity result in the linear setting is the **De Giorgi-Nash-Moser theory**; it says that weak solutions to

$$\text{div}(A(x)Du) = 0 \quad \text{in} \quad B_1$$

are of Hölder continuous, provided the matrix A is well-prepared.

When entering the realm of nonlinear PDEs, we start by revisiting the former results. This is the content of the Evans-Krylov and Krylov-Safonov theories.

Let $F : \mathcal{S}(d) \to \mathbb{R}$ be a fully nonlinear (λ, Λ)-elliptic operator; that is, to satisfy

$$\lambda\|N\| \leq F(M+N) - F(M) \leq \Lambda\|N\|,$$

for every $M,\, N \in \mathcal{S}(d)$ with $N \geq 0$.

The **Evans-Krylov theory** states that (viscosity) solutions to

$$F(D^2u) = 0 \quad \text{in} \quad B_1 \tag{1}$$

are in $\mathcal{C}^{2,\alpha}_{loc}(B_1)$, provided F **is convex**.

When the convexity of the operator F fails, the **Krylov-Safonov theory** assures that solutions to (1) are in $\mathcal{C}^{1,\alpha}_{loc}(B_1)$.

A fundamental question haunted the community for about twenty years though: would solutions to (1) be mor regular than stated by the Krylov-Safonov, in the absence of convexity?

Set in negative recently by **Nadirashvili and Vladut**, this issue launched a new direction of research in the area: what are the structures capable of unveil further regularity?

Inspired by intrinsically geometric ideas due to L. Caffarelli – and reported in a trail-blazing paper in the *Ann. Math.*, 1989 – we investigate work in line with this research agenda through a class of techniques referred to as **regularity transmission by approximation methods**.

Regularity transmission by approximation methods

Given a PDE of interest, we related it with an auxiliary one, for which a richer theory is available. The strategy to connect both problems is by means of an approximation method.

For instance, the equation

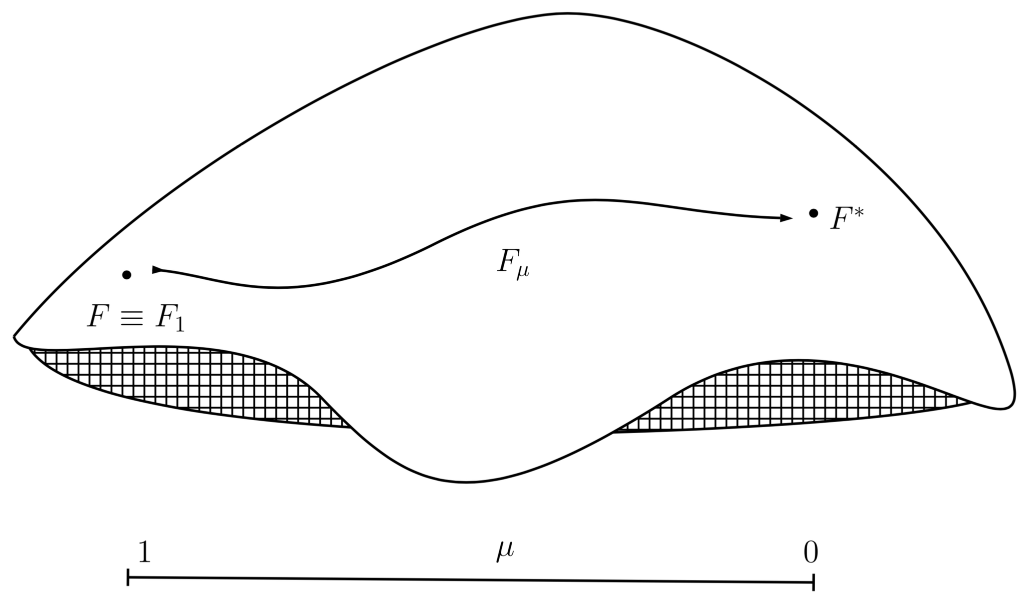
$$F(D^2u) = f \quad \text{in} \quad B_1$$

can be studied through the regularity theory available for (1), provided f satisfies certain (fairly general) conditions.

Another example has to do with diffusion operators of p-Laplacian type. I.e., we have recently discovered that information on the solutions to

$$\Delta_p u := \text{div}(|Du|^{p-2}Du) = f \quad \text{in} \quad B_1$$

can be accessed through the analysis of the Laplace equation in case p is close to 2 in some sense that can be made precise.



The main difficulty is to design the most **robust path connecting** both problems. In what follows we give two examples

A nonconvex model: the Isaacs equation

We are interested in the following (fully nonlinear) PDE:

$$\sup_{\alpha\in\mathcal{A}}\inf_{\beta\in\mathcal{B}}\left[-\operatorname{Tr}\left(A_{\alpha,\beta}(x)D^2u\right)\right] = f \quad \text{in} \quad B_1.$$

We suppose:

1. Ellipticity: the matrix $A_{\alpha,\beta} : B_1 \times \mathcal{A} \times \mathcal{B} \to \mathbb{R}^{d^2}$ satisfies
$$\lambda Id \leq A_{\alpha,\beta} \leq \Lambda Id,$$
for $0 < \lambda \leq \Lambda$, fixed;
2. the source term $f : B_1 \to \mathbb{R}$ is in $p - BMO(B_1)$;
3. $\mathcal{A}$ and $\mathcal{B}$ are compact metric spaces.

Isaacs equation arises in the context of two-players, zero-sum, differential games, as introduced by Rufus Isaacs;

→ Applications to life and social sciences;

Example of an operator that is neither convex nor concave;

→ Regularity: Evans-Krylov theory is not available.

We propose an **approximation mechanism based on the (homogeneous) Bellman equation**:

$$\inf_{\beta\in\mathcal{B}}\left[-\operatorname{Tr}\left(A_\beta(x)D^2h\right)\right] = 0 \quad \text{in} \quad B_1,$$

where $A_\beta : B_1 \times \mathcal{B} \to \mathbb{R}^{d^2}$ is a (λ, Λ)-elliptic matrix.

At the core of the method is a (uniform) smallness regime of the form

$$|A_{\alpha,\beta}(x) - A_\beta(x)| \ll 1/2.$$

Ideally, different smallness regimes yield different estimates, in Sobolev and Hölder spaces.

Arising in the theory of (stochastic) optimal control, the Bellman equation is convex with respect to the Hessian;

⟹ The Evans-Krylov theory is available and solutions to

$$\inf_{\beta\in\mathcal{B}}\left[-\operatorname{Tr}\left(A_\beta(x_0)D^2h\right)\right] = 0 \quad \text{in} \quad B_1,$$

have $\mathcal{C}^{2,\gamma}$-estimates, for every $x_0 \in B_1$.

GOAL: to import regularity from the Bellman equation – limiting profile – to the Isaacs equation – original problem.

Our main result reads as follows:

Theorem (P., Ann. Inst. H. Poincaré – Anal. NL, 19) Let $u \in \mathcal{C}(B_1)$ be a viscosity solution to the Isaacs equation. Suppose

$$\sup_{B_r}|A_{\alpha,\beta}(x) - A_\beta| \leq \varepsilon r^\gamma,$$

uniformly, and

$$\left(\int B_r\,|f(x)|^p\,dx\right)^{\frac{1}{p}} \leq \varepsilon r^\gamma.$$

1. Then, u is of class $\mathcal{C}^{2,\gamma}$ at the origin;
2. if $f \equiv 0$, we have $u \in \mathcal{C}^{2,\gamma}_{loc}(B_1)$ with
$$\|u\|_{\mathcal{C}^{2,\gamma}(B_{1/2})} \leq C\,\|u\|_{L^\infty(B_1)}\,.$$

The proof of this result are amounts to establish the existence of a sequence of polynomials $(P_k)_{k\in\mathbb{N}}$,

$$P_k(x) = a_k + b_k\cdot x + \frac{1}{2}x^TC_kx,$$

and $\rho \ll 1/2$, satisfying

$$\inf_{\beta\in\mathcal{B}}\left[-\operatorname{Tr}\overline{A}_\beta C_k\right] = 0;$$
$$\|u - P_k\|_{L^\infty(B_{\rho^k})} \leq \rho^{k(2+\gamma)}$$

and

$$|a_k - a_{k-1}| + \rho^{k-1}|b_k - b_{k-1}| + \rho^{2(k-1)}|C_k - C_{k-1}| \leq C\rho^{(k-1)(2+\gamma)},$$

with $P_0 \equiv P_{-1} \equiv 0$. It ensures oscillation control at the discrete level.

To obtain such a sequence of polynomials we start with an **Approximation Lemma**, which is further iterated.

Proposition (Approximation Lemma) Let $u \in \mathcal{C}(B_1)$ be a viscosity solution to the Isaacs equation. For every$\delta > 0$, there exists $\varepsilon > 0$ such that, if

$$\sup_{B_r}|A_{\alpha,\beta}(x) - A_\beta| \leq \varepsilon r^\gamma,$$

uniformly, and

$$\left(\int B_r\,|f(x)|^p\,dx\right)^{\frac{1}{p}} \leq \varepsilon r^\gamma,$$

then, there exists $h \in \mathcal{C}^{2,\overline{\gamma}}(B_{7/8})$ satisfying

$$\|u - h\|_{L^\infty(B_{7/8})} \leq \delta.$$

Fully nonlinear degenerate diffusions

Here, we examine

$$|Du|^{\theta(x)}F(D^2u) = f \quad \text{in} \quad B_1,$$

where

1. ((λ, Λ)-Ellipticity) the operator $F : S(d) \to \mathbb{R}$ satisfies
$$\lambda\|N\| \leq F(M+N) - F(M) \leq \Lambda\|N\|,$$
for every $M,\, N \in S(d)$ with $N \geq 0$, for some $0 < \lambda \leq \Lambda$ fixed;
2. (Integrability of the source term) the source term satisfies $f \in L^\infty(B_1) \cap \mathcal{C}(B_1)$;
3. (Variable exponent) the exponent $\theta : B_1 \to \mathbb{R}$ is merely bounded and measurable. In particular, $-1 < \theta(\cdot)$.

This problem can be addressed as the nonvariational counterpart of the p-Laplacian, in the presence of state-dependent degeneracy rates. Among its applications, we highlight image-processing and the study of functional spaces with weights. In the case of constante exponents $\theta(\cdot) \equiv \theta$, there is a number of previous results available. In particular:

⟶ Previously studied by Araújo, Ricarte and Teixeira, Birindelli and Demengel, Imbert and Silvestre;

⟶ Local and global regularity, comparison principles and information on (appropriate notions of) eigenvalues;

⟶ The optimal regularity of the solutions is $\mathcal{C}^{1,\alpha^*}$, with

$$\alpha^* := \min\left\{\alpha_0, \frac{1}{1+\theta}\right\}.$$

We have obtained the following regularity result:

Theorem (Bronzi, P., Rampasso, Teixeira)

Let $u \in \mathcal{C}(B_1)$ be a viscosity solution to

$$|Du|^{\theta(x)}F(D^2u) = f \quad \text{in} \quad B_1.$$

Then, $u \in \mathcal{C}^{1,\alpha}_{loc}(B_1)$, where

$$\alpha := \min\left\{\alpha_0, \frac{1}{1 + \|\theta^+\| + \|\theta^-\|}\right\}.$$

Morevoer, there exists $C > 0$, universal, such that

$$\|u\|_{\mathcal{C}^{1,\alpha}(B_{1/2})} \leq C\left(1 + \|u\|_{L^\infty(B_1)} + \|f\|_{L^\infty(B_1)}^{\frac{1}{1+\inf\theta}}\right)$$

We stress the following:

1. Our arguments accommodate (a continuous approximation of) exponents of the form
$$\theta(x) := \frac{1}{2}\sin\left(\frac{1}{|x|}\right)$$
2. The method seamlessly addresses the degenerate and singular regimes;
3. Explicit dependence of the regularity class on both regimes.

The strategy of the proof lies on the following structures:

1. First, we import $\mathcal{C}^{1,\alpha}$-regularity from the homogeneous, uniformly elliptic, equation governed by F,
$$F(D^2u) = 0 \quad \text{in} \quad B_1.$$
2. Compactness of the solutions: an application of Jensen's Lemma yields $u \in \mathcal{C}^\beta(B_1)$, uniformly;
3. Approximation Lemma;
4. Iterative argument.

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