An approach of the linear heat equation in spaces based on the Fourier transform

Cláudia Aline A. S. Mesquita[1] & Lucas Catão de F. Ferreira[2]

[1] Instituto de Ciência e Tecnologia - UNIFESP
[2] Instituto de Matemática, Estatística e Computação Científica - UNICAMP

caas.mesquita@unifesp.br, lcff@ime.unicamp.br

UNIFESP
UNIVERSIDADE FEDERAL DE SÃO PAULO
1933

Introduction

In this work we investigate the heat equation with potential:

$$\begin{cases} u_t - \Delta u - V(x)u = 0, \text{ in } \mathbb{R}^n, \\ u(x,0) = u_0(x), \quad \text{in } \mathbb{R}^n \end{cases} \tag{1}$$

with $n \geq 3$ and V is a critical multipolar potential. An example of this type of potential is the potential of Hardy:

$$V(x) = \frac{\lambda}{|x|^2} \tag{2}$$

and multipole versions

$$V(x) = \sum_{j=1}^{l} \frac{\lambda_j}{|x - x^j|^2} \text{ ou } V(x) = \sum_{j=1}^{l} \frac{dj(x - x^j)}{|x - x^j|^3} \tag{3}$$

with $x^j = (x_1^j, x_2^j, ..., x_n^j) \in \mathbb{R}^n$ and $d^j = (d_1^j, d_2^j, ..., d_n^j) \in \mathbb{R}^n$ are constant vectors.

Motivations

An important work due to Baras and Goldstein (see references) establishes a threshold for the existence (or not) of solution positive in $L^2(\mathbb{R}^n)$, for the problem (1) with V being Hardy's potential. More precisely,

- Has been proved a result of existence of solution to (1) in $L^2(\mathbb{R}^n)$) when $0 \leq \lambda \leq \lambda^*$;
- For $\lambda < \lambda*$ has been proved a result of no solution exists.

being $\lambda* = \dfrac{(n-2)^2}{4}$ the best constant of inequality by Hardy:

$$\lambda * \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} \leq ||\nabla u||^2_{L^2(\mathbb{R}^n)}$$

In this and many other later works, is the use of this inequality that imposes that the solutions u are in $L^2(\mathbb{R}^n)$, condition $0 \leq \lambda \leq \lambda^*$ for good placement of solutions in this space. With this, an issue natural arises:

There is a space other than $L^2(\mathbb{R}^n)$ in which the problem (1), where V is the potential of Hardy, well placed for $0 \leq \lambda \leq \lambda^*$?

In this paper we investigate this question using the PM^k-spaces and a strategy based in the Fourier transform that does not use the inequality of Hardy.

PM^k - Spaces

For every $0 < k < n$ the PM^k space is defined by:

$$PM^k = \{u \in \mathcal{S}' : \hat{u} \in L^1_{loc}(\mathbb{R}^n), ||u||_{PM^k} < +\infty\}$$

where $||u||_{PM^k} = ess\ sup_{\xi \in \mathbb{R}^n} |\xi|^k |\mathcal{F}[u](\xi)| < \infty$ defines a norm, with which this is a space of Banach.

Integral formulation

The problem (1) is formally equivalent the following functional equation:

$$u(t) = G(t)u_0 + L_V u(t), \tag{4}$$

where operators are defined using transform by:

$$\mathcal{F}[G(t)u_0](\xi, t) = e^{-4\pi^2|\xi|^2 t}\mathcal{F}[u_0](\xi)$$
$$\mathcal{F}[L_V u(t)](\xi) = \int_0^t e^{-4\pi^2|\xi|^2(t-s)} \left(\mathcal{F}[V] * \mathcal{F}[u]\right)(\xi, s)ds.$$

Equality in (4) must be understood in the sense of tempered distributions.

Main result obtained

The flow of (1) will be studied in the following space:

$$X_k = BC_w([0,\infty); PM^k).$$

Theorem 1: Suppose that $V \in PM^{n-2}$ and $u_0 \in PM^k$ with $2 < k < n$.

- (Existence and uniqueness) Let $C_{n-2,k} = \dfrac{K(2, n-k)}{4\pi^2}$ and assume that
$$||V||_{PM^{n-2}} = \frac{1}{C_{n-2,k}},$$
Then the functional equation (4) has a unique solution u in X_k.
- (Continuous dependence) The data-solution map $(u_0, V) \to u$ is Lipschitz continuous from $PM^k \times PM^{n-2} \to X_k$.

Remark 1: (Hardy potential) For $V(x) = \frac{\lambda}{|x|^2}$, the condition on V becomes equivalent to $|\lambda| < (k-2)(n-k)$. Notice that the maximum of $(k-2)(n-k)$ is $\lambda* = \dfrac{(n-2)^2}{4}$, which is reached at $k = \frac{n+2}{2}$. Then the item (i) provides a global solution u for (4) for all $u_0 \in PM^{1+n/2}$ and $0 \leq \lambda \leq \lambda*$.

We also showed results of autosimilarity and behavior asymptotic.

Conclusion

- We obtained a positive response to the initial question, that is, we show that (1) is well placed in the PM^k spaces for $0 \leq \lambda < \lambda*$ using a strategy based on in the Fourier transform not uses Hardy's inequality;
- Since there is no any inclusion relation between L^2 and PM^k, our results indicate that $\lambda*$ is intrinsic of the PDE and independent of a particular approach;
- As far as we know, our results constitute a first example of use approach to address a problem of global existence for an EDP with an optimum threshold value by another technique.

***Agradecemos ao CNPq pelo apoio financeiro.**

References

[1] L. C. F Ferreira and C. A. A. S. Mesquita. An approach without using Hardy inequality for the linear heat equation with singular potential. Communications in Contemporary Mathematics. V. 17, p. 1550041, 2015.

[2] P. Baras and J. A. Goldstein. The heat equation with a singular potential. Transactions of the American Mathematical Society, 1984.