Split Holomorphic Distributions on Fano Threefolds

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Introduction

In this work, we study holomorphic distributions on a smooth weighted projective complete intersection Fano threefold X with Picard number equal to one.

Fano threefolds with rank one Picard group have been classified by Iskovskih and Mukai. The *index* of X is the largest integer ι_X such that the canonical line bundle K_X is divisible by ι_X in $\mathrm{Pic}(X)$.

In [4], Kobayashi and Ochiai showed that the index ι_X is at most $\dim(X)+1$ and $\iota_X = \dim(X)+1$, if and only if $X \simeq \mathbb{P}^n$. Moreover, $\iota_X = \dim(X)$, if and only if $X \simeq Q^n \subset \mathbb{P}^{n+1}$, where Q^n is a smooth quadric.

Objectives

1. Characterization of distributions whose tangent sheaf and conormal sheaf are arithmetically Cohen Macaulay (aCM), i.e. has no intermediate cohomology;
2. To study algebro-geometric properties of singular schemes of distributions.

Holomorphic Distributions

Let X be a smooth complex manifold.

(i) A **codimension k distribution** $\mathfrak{F}$ on X is given by an exact sequence

$$\mathfrak{F}:\ 0 \longrightarrow T_{\mathfrak{F}} \xrightarrow{\ \phi\ } TX \xrightarrow{\ \pi\ } N_{\mathfrak{F}} \longrightarrow 0, \qquad (1)$$

where $T_{\mathfrak{F}}$ is a coherent sheaf of rank $r_{\mathfrak{F}} := \dim(X) - k$, and $N_{\mathfrak{F}} := TX/\phi(T_{\mathfrak{F}})$ is a torsion free sheaf.

(ii) The sheaves $T_{\mathfrak{F}}$ and $N_{\mathfrak{F}}$ are called the **tangent** and the **normal** sheaves of $\mathfrak{F}$, respectively.

(iii) $\mathrm{Sing}(\mathfrak{F}) = \{x \in X \mid (N_{\mathfrak{F}})_x$ is not a free $\mathcal{O}_{X,x}-$ module$\}$ is the **singular set** of the distribution $\mathfrak{F}$.

⋆ A distribution $\mathfrak{F}$ is said to be **locally free** if $T_{\mathfrak{F}}$ is a locally free sheaf.

Fano threefolds with rank one Picard group

A Fano 3-fold X can have $\iota_X \in \{1,2,3,4\}$.

When $\iota_X = 4$, we have $X \simeq \mathbb{P}^3$. When $\iota_X = 3$, we have $X \simeq Q^3$. When $\iota_X = 2$, the variety X is called a del Pezzo Fano threefold and when $\iota_X = 1$, the variety X is called a Prime Fano threefold.

Theorem [Giraldo, Pan Collantes]: The tangent sheaf of a foliation of dimension 2 on $\mathbb{P}^3$ splits if and only if $Z := \mathrm{Sing}(\mathfrak{F})$ is an arithmetically Cohen-Macaulay curve.

Theorem [Corrêa, Jardim, Vidal]: Let $\mathfrak{F}$ be a distribution on $\mathbb{P}^n$ of codimension k, such that the tangent sheaf $T_{\mathfrak{F}}$ is locally free and whose singular locus has the expected dimension $n-k-1$. If $T_{\mathfrak{F}}$ splits as a sum of line bundles, then $\mathrm{Sing}(\mathfrak{F})$ is arithmetically Cohen–Macaulay. Conversely, if $k=1$ and $\mathrm{Sing}(\mathfrak{F})$ is arithmetically Cohen–Macaulay, then $T_{\mathfrak{F}}$ splits as a sum of line bundles.

Results

Theorem 1: Let $\mathfrak{F}$ be a distribution of codimension one on a smooth weighted projective complete intersection Fano threefold X, such that the tangent sheaf $T_{\mathfrak{F}}$ is locally free and $\det(N_{\mathfrak{F}}) = \mathcal{O}_X(r)$, for some $r > 0$.

(i) If $X = Q^3$ and $T_{\mathfrak{F}}$ either splits as a sum of line bundles or is a spinor bundle, then Z is arithmetically Buchsbaum, with $h^1(Q^3, I_Z(r-2)) = 1$ being the only nonzero intermediate cohomology for $H^i(I_Z)$. Conversely, if Z is arithmetically Buchsbaum with $h^1(Q^3, I_Z(r-2)) = 1$ being the only nonzero intermediate cohomology for $H^i(I_Z)$ and $h^2(T_{\mathfrak{F}}(-2)) = h^2(T_{\mathfrak{F}}(-1-c_1(T_{\mathfrak{F}}))) = 0$, then $T_{\mathfrak{F}}$ either split or is a spinor bundle.

(ii) If $\iota_X = 2$ and $T_{\mathfrak{F}}$ has no intermediate cohomology, then $H^1(X, I_Z(r+t)) = 0$ for $t < -6$ and $t > 8$. Conversely, if $H^1(X, I_Z(r+t)) = 0$ for $t < -6$ and $t > 8$, and $H^2(X, T_{\mathfrak{F}}(t)) = 0$ for $t \leq 8$ and $H^1(X, T_{\mathfrak{F}}(s)) = 0$ for $s \neq -t - \iota_X - c_1(T_{\mathfrak{F}})$, then $T_{\mathfrak{F}}$ has no intermediate cohomology.

(iii) If $\iota_X = 1$ and $T_{\mathfrak{F}}$ has no intermediate cohomology, then $H^1(X, I_Z(r+t)) = 0$ for $t < -4$ and $t > 4$. Conversely, if $H^1(X, I_Z(r+t)) = 0$ for $t < -4$ and $t > 4$, and $H^2(X, T_{\mathfrak{F}}(t)) = 0$ for $t \leq 4$ and $H^1(X, T_{\mathfrak{F}}(s)) = 0$ for $s \neq -t - \iota_X - c_1(T_{\mathfrak{F}})$, then $T_{\mathfrak{F}}$ has no intermediate cohomology.

Example: Let S be the spinor bundle on Q^3. Then $S(1-t)$ is the tangent sheaf of a codimension one distribution $\mathfrak{F}$, for all $t \geq 0$

Theorem 2: Let $\mathfrak{F}$ be a distribution of dimension one on a smooth weighted projective complete intersection Fano threefold such that $N_{\mathfrak{F}}^*$ is locally free and $\det(N_{\mathfrak{F}}) = \mathcal{O}_X(r)$, for some $r > 0$.

(i) If $N_{\mathfrak{F}}^*$ is arithmetically Cohen Macaulay and $\iota_X \in \{1,2,3,4\}$, then Z is arithmetically Buchsbaum, with $h^1(X, I_Z(r)) = 1$ being the only nonzero intermediate cohomology for $H^i(I_Z)$.

(ii) If Z is arithmetically Buchsbaum with $h^1(X, I_Z(r)) = 1$ being the only nonzero intermediate cohomology for $H^i(I_Z)$ and $h^2(N_{\mathfrak{F}}^*) = h^2(N_{\mathfrak{F}}^*(-c_1(N_{\mathfrak{F}}^*) - \iota_X)) = 0$ and $\iota_X \in \{1,2,3\}$, then $N_{\mathfrak{F}}^*$ is arithmetically Cohen Macaulay.

Theorem 3: Let $\mathfrak{F}$ be a codimension one distribution with singular scheme Z and let X be a smooth weighted projective complete intersection Fano 3-fold. If $h^2(T_{\mathfrak{F}}(-r)) = 0$ and $C \subset X$, $C \neq \emptyset$, then Z is connected and of pure dimension 1, and $T_{\mathfrak{F}}$ is locally free. Conversely, for $r \neq \iota_X$, if $Z = C$ is connected, then $T_{\mathfrak{F}}$ is locally free and $h^2(T_{\mathfrak{F}}(-r)) = 0$.

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