Minimal $*$-varieties and minimal supervarieties of polynomial growth

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Abstract

By a φ-variety $\mathcal{V}$, we mean a supervariety or a $*$-variety generated by an algebra over a field F of characteristic zero.

For $\mathcal{V} = \text{var}^{\varphi}(A)$ a φ-variety of algebras, we define $c_n^{\varphi}(\mathcal{V}) = c_n^{\varphi}(A)$ and we say that $\mathcal{V}$ is **minimal of polynomial growth** n^k if $c_n^{\varphi}(\mathcal{V})$ grows like n^k, but $c_n^{\varphi}(\mathcal{U})$ grows like n^t with $t < k$, for any proper φ-subvariety $\mathcal{U}$ of $\mathcal{V}$

Motivated by the results in [1], in this work, we characterize the minimal φ-varieties generated by unitary algebras of growth n^k. We prove that for $k \leq 2$ there are only a finite number of algebras generating such minimal φ-varieties. For $k \geq 3$, we show that the number of minimal φ-varieties can be infinity and give a receipt for the construction of their T^{φ}-ideals.

Characterization of minimal φ-varieties

Now let's consider the unitary $*$-algebras generating minimal varieties of polynomial growth given by La Mattina and Martino in [4].

For any $k \geq 2$, consider the matrix $E = \sum_{i=2}^{k-1} e_{i,i+1} + e_{2k-i,2k-i+1} \in UT_{2k}$ and the following subalgebras of UT_{2k} :

$P_{k,\rho} = \text{span}_F\{I_{2k}, E, \ldots, E^{k-2}; e_{12} - e_{2k-1,2k}, e_{13}, \ldots, e_{1k}, e_{k+1,2k}, e_{k+2,2k}, \ldots, e_{2k-2,2k}\}$ e

$Q_{k,\rho} = \text{span}_F\{I_{2k}, E, \ldots, E^{k-2}; e_{12} + e_{2k-1,2k}, e_{13}, \ldots, e_{1k}, e_{k+1,2k}, e_{k+2,2k}, \ldots, e_{2k-2,2k}\}$, endowed with the reflection involution ρ, where I_{2k} denotes the identity matrix of order $2k$.

The authors also considered the following commutative subalgebra of UT_k $C_k = \left\{\alpha I_k + \sum_{1\leq i<k} \alpha_i E_1^i \,:\, \alpha, \alpha_i \in F\right\}$, for $k \geq 2$, where $E_1 = \sum_{i=1}^{k-1} e_{i,i+1}$ and I_k denotes the identity matrix of order k. We denote by $C_{k,*}$ the algebra C_k with involution given by $(\alpha I_k + \sum_{1\leq i<k} \alpha_i E_1^i)^* = \alpha I_k + \sum_{1\leq i<k} (-1)^i \alpha_i E_1^i$.

Furthermore, in [3], La Mattina, presented some superalgebras generating minimal varieties. For $k \geq 2$, let C_k^{gr} to be the algebra C_k with elementary grading induced by $g = (0, 1, 0, 1, ...) \in \mathbb{Z}_2^k$.

For some $k > 2$, let $N_k = \text{span}_F\{I_k, E, \ldots, E^{k-2}; e_{12}, e_{13}, ..., e_{1k}\}$ the subalgebra of UT_k, where $E = \sum_{i=1}^{k-1} e_{i,i+1} \in UT_k$ and I_k denotes the $k \times k$ identity matrix.

Consider the superalgebra N_k^{gr} to be the algebra N_k with elementar $\mathbb{Z}_2$-grading induced by $g = (0, 1, ..., 1) \in \mathbb{Z}_2^k$.

For $k \geq 2$, consider G_k to be the finite dimensional Grassmann algebra with 1 generated by the elements $e_1, ..., e_k$ over F subject to the condition $e_i e_j = -e_j e_i$, $1 \leq i, j \leq k$. Finally consider G_k^{gr} to be the finite dimensional Grassmann algebra G_k endowed with the grading induced by the canonical grading of $\mathcal{G}^{gr}$.

In what follows we will consider $\mathcal{V}$ be a φ-variety of polynomial growth generated by a unitary algebra.

Theorem 1. *Let $\mathcal{V}$ be a φ-variety of polynomial growth n^k and let $\pi_n^{\varphi}(\mathcal{V}) = \sum_{(\gamma,\sigma)\vdash n} \widetilde{m}_{\gamma,\sigma}\chi_{\gamma,\sigma}$ be its nth proper φ-cocharacter. Then $\mathcal{V}$ is minimal if and only if the following hold*
$i)$ There exists a pair of partitions $(\lambda, \mu) \vdash k$, such that $\widetilde{m}_{\lambda,\mu} = 1$ and for all $(\nu, \theta) \vdash k$ with $(\nu, \theta) \neq (\lambda, \mu)$ we have $\widetilde{m}_{\nu,\theta} = 0$;
$ii)$ if $(\lambda, \mu) \vdash k$ is a pair of partitions and $f_{\lambda,\mu}$ is a Y-proper highest weight vector such that $f_{\lambda,\mu} \notin Id^{\varphi}(\mathcal{V})$, then for each $h < k$ and for each $(\nu, \theta) \vdash h$, if $f_{\nu,\theta}$ is a Y-proper highest weight vector such that $f_{\nu,\theta} \notin Id^{\varphi}(\mathcal{V})$, we must have $f_{\nu,\theta} \rightsquigarrow f_{\lambda,\mu} \pmod{Id^{\varphi}(\mathcal{V})}$, this is, $f_{\lambda,\mu} \in \langle Id^{\varphi}(\mathcal{V}), f_{\nu,\theta}\rangle_{T^{\varphi}}$.

By using our theorem we can give new examples of minimal φ-varieties.

Theorem 2. *Let $\mathcal{T}_{k,*}$ be a $*$-variety generated by a unitary algebra and determined by T^*-ideal*

$$\langle [y_1, y_2], [y, z], z_1 \circ z_2, z_1 z_2 \ldots z_{k+1}\rangle_{T^*}.$$

Then $\mathcal{T}_{k,*}$ is minimal of growth n^k.

Denote by $G_{k,*}$ the algebra G_k endowed with the involution $*$ such that $e_i^* = -e_i$, $i = 1, 2, \ldots, k$.

La Mattina and Misso (2006) proved that

$$\text{Id}^*(G_{2,*}) = \langle [y_1, y_2], [y, z], z_1 \circ z_2, z_1 z_2 z_3\rangle_{T^*}.$$

Notice that $\mathcal{T}_{2,*} = \text{var}^*(G_{2,*})$, that is, $G_{2,*}$ is a minimal $*$-algebra of quadratic growth. Furthermore

- $\text{Id}^*(G_{3,*}) = \langle [y_1, y_2], [y, z], [z_1, z_2 z_3], z_1 z_2 z_3 + z_3 z_2 z_1, z_1 z_2 z_3 z_4\rangle_{T^*}$;
- $\mathcal{U} = \text{var}^*(G_{3,*}) \Rightarrow \mathcal{T}_{3,*} \subsetneq \mathcal{U}$;
- $G_{3,*}$ has cubic growth and it is not minimal.

Minimal φ-varieties with at most quadratic growth

In order to start our classification, it is important to know the decomposition of the character of the space Γ_n^{φ} of multilinear Y-proper polynomials of degree n, for $n = 2, 3$.

Theorem 3. $i)\chi(\Gamma_2^{\varphi}) = \chi_{(1),(1)} + \chi_{(1^2),\emptyset} + \chi_{\emptyset,(1^2)} + \chi_{\emptyset,(2)}$;
$ii)\chi(\Gamma_3^{\varphi}) = \chi_{(2,1),\emptyset} + \chi_{(2),(1)} + 2\chi_{(1^2),(1)} + 2\chi_{(1),(1^2)} + 2\chi_{(1),(2)} + \chi_{\emptyset,(1^3)} + 2\chi_{\emptyset,(2,1)} + \chi_{\emptyset,(3)}$.

Consider the multilinear Y-proper polynomials of degree 2: $g_1 = [y, z], g_2 = [y_1, y_2], g_3 = [z_1, z_2], g_4 = z_1 \circ z_2$ and the multilinear Y-proper polynomials of degree 3: $f_1 = [y_1, y_2, y_3] + [y_3, y_2, y_1]$, $f_2 = [y_1, z, y_2] + [y_2, z, y_1]$, $f_3 = z[y_1, y_2]$ and $f_3' = [y_1, y_2, z]$, $f_4 = z_1[y, z_2]$ and $f_4' = [z_1, z_2, y]$, $f_5 = z_1[y, z_2] + z_2[y, z_1]$ and $f_5' = [y, z_1, z_2] + [y, z_2, z_1]$, $f_6 = St_3(z_1, z_2, z_3)$, $f_7 = z_1[z_3, z_2] + z_3[z_1, z_2]$ and $f_7' = [z_1, z_2, z_3] + [z_3, z_2, z_1]$, $f_8 = \sum_{\sigma\in S_3} z_{\sigma(1)} z_{\sigma(2)} z_{\sigma(3)}$.

Theorem 4. *Let A be a unitary $*$-algebra such that $c_n^*(A) \leq kn^2$, for some nonzero constant k. Then A generates a minimal $*$-variety if and only if either $A \sim_{T^*} C_{2,*}$ or $A \sim_{T^*} P_{3,\rho}$ or $A \sim_{T^*} Q_{3,\rho}$ or $A \sim_{T^*} G_{2,*}$ or $A \sim_{T^*} C_{3,*}$.*

Theorem 5. *Let A be a unitary superalgebra such that $c_n^{gr}(A) \leq kn^2$, for some nonzero constant k. Then A generates a minimal supervariety if and only if either $A \sim_{T_2} C_2^{gr}$ or $A \sim_{T_2} N_3^{gr}$ or $A \sim_{T_2} G_2$ or $A \sim_{T_2} G_2^{gr}$ or $A \sim_{T_2} C_3^{gr}$.*

Minimal φ-varieties of cubic growth

Let L_* be a unitary $*$-algebra generating the $*$-variety $\mathcal{T}_{3,*}$.

Theorem 6. *Let A be a unitary $*$-algebra such that $\pi_3^*(A) = \chi_{\lambda,\mu}$ where $m_{\lambda,\mu} = 1$ in the decomposition of $\chi(\Gamma_3^{\varphi})$. Then A generates a minimal $*$-variety of cubic growth if and only if either $A \sim_{T^*} Q_{4,\rho}$ or $A \sim_{T^*} P_{4,\rho}$ or $A \sim_{T^*} C_{4,*}$ or $A \sim_{T^*} L_*$.*

Theorem 7. *Let A be a unitary superalgebra such that $\pi_3^{gr}(A) = \chi_{\lambda,\mu}$ where $m_{\lambda,\mu} = 1$ in the decomposition of $\chi(\Gamma_3^{\varphi})$. Then A generates a minimal supervariety of cubic growth if and only if either $A \sim_{T_2} N_4$ or $A \sim_{T_2} N_4^{gr}$ or $A \sim_{T_2} G_3^{gr}$ or $A \sim_{T_2} C_4^{gr}$.*

In what follows we consider $\mathcal{V}$ to be a φ-variety of cubic growth such that $\pi_3^{\varphi}(\mathcal{V}) = \chi_{\lambda,\mu}$, where $\chi_{\lambda,\mu}$ is one of the four irreducible characters with $m_{\lambda,\mu} = 2$ in the decomposition of $\chi(\Gamma_3^{\varphi})$.

Let h and $\tilde{h}$ be Y-proper linearly independent highest weight vectors associated to (λ, μ). Then h and $\tilde{h}$ are linearly dependent modulo $\text{Id}^{\varphi}(\mathcal{V})$. We prove that $\mathcal{V}$ is uniquely determined, up to a scalar γ in F, by a linear combination $\tilde{h} + \gamma h$.

In this case, we will denote the T^{φ}-ideal of $\mathcal{V}$ by $I_{\tilde{h}+\gamma h}$.

Proposition 1. *There is no $*$-variety $\mathcal{V}$ of cubic growth such that $\pi_3^*(\mathcal{V}) = \chi_{(1),(2)}$.*

For the pair of partition $(\lambda, \mu) \vdash 3$ fixed as above, let $Q^{\lambda,\mu} = \{f_{\sigma,\tau} \,:\, (\sigma, \tau) \vdash 3, (\sigma, \tau) \neq (\lambda, \mu)\}$.

Theorem 8. *Let $\pi_3^*(\mathcal{V}) = \chi_{\varnothing,(2,1)}$. Then $\mathcal{V}$ is a minimal $*$-variety of growth n^3 if and only if $Id^*(\mathcal{V})$ coincides with one of the following T^*-ideals*
$i)$ $I_{f_7+2f_7'} = \left\langle \Gamma_4^*, Q^{\varnothing,(2,1)}, g_1, g_2, g_4, f_7 + 2f_7'\right\rangle_{T^*}$;
$ii)$ $I_{f_7+\gamma f_7'} = \left\langle \Gamma_4^*, Q^{\varnothing,(2,1)}, g_2, f_7 + \gamma f_7'\right\rangle_{T^*}$, where $\gamma \in F$ is any scalar such that $\gamma \neq 2$.

Theorem 9. *Let $\pi_3^{gr}(\mathcal{V}) = \chi_{(1),(2)}$. Then $\mathcal{V}$ is a minimal supervariety of growth n^3 if and only if $Id^{gr}(\mathcal{V})$ coincides with the T_2-ideal $I_{f_5'} = \left\langle \Gamma_4^{gr}, Q^{(1),(2)}, f_5'\right\rangle_{T_2}$.*

Theorem 10. *Let $\pi_3^{gr}(\mathcal{V}) = \chi_{\varnothing,(2,1)}$. Then $\mathcal{V}$ is a minimal supervariety of growth n^3 if and only if $Id^{gr}(\mathcal{V})$ coincides with one of the T_2-ideals*
$i)$ $I_{f_7'} = \left\langle \Gamma_4^{gr}, Q^{\varnothing,(2,1)}, g_2, f_7'\right\rangle_{T_2}$;
$ii)$ $I_{f_7+2f_7'} = \left\langle \Gamma_4^{gr}, Q^{\varnothing,(2,1)}, g_2, g_4, f_7 + 2f_7'\right\rangle_{T_2}$;
$iii)$ $I_{f_7+\gamma f_7'} = \left\langle \Gamma_4^{gr}, Q^{\varnothing,(2,1)}, g_2, f_7 + \gamma f_7'\right\rangle_{T_2}$, where $\gamma \in F$ is any scalar such that $\gamma \neq 2$.

In [2], we generalize the procedure above in order to classify the minimal φ-varieties generated by unitary algebras with growth n^k, $k \geq 4$ and we provide a method to construct their T^{φ}-ideals.

Minimal φ-varieties of growth $n^k, k \geq 4$

Consider the decomposition of the H_k-character of Γ_k^{φ}

$$\chi(\Gamma_k^{\varphi}) = \sum_{(\lambda,\mu)\vdash k} \overline{m}_{\lambda,\mu}\chi_{\lambda,\mu}.$$

For each pair of partitions $(\lambda, \mu) \vdash k$, let

- $f_{\lambda,\mu}^1, f_{\lambda,\mu}^2, \cdots, f_{\lambda,\mu}^{\overline{m}_{\lambda,\mu}}$ be Y-proper linearly independent highest weight vectors associated to (λ, μ);
- $V^{\lambda,\mu} = \text{span}_F\{f_{\lambda,\mu}^1, f_{\lambda,\mu}^2, \cdots, f_{\lambda,\mu}^{\overline{m}_{\lambda,\mu}}\}$.

Proposition 2. *Let $\mathcal{V}$ be a φ-variety generated by a unitary algebra and consider*

$$\pi_k^{\varphi}(\mathcal{V}) = \sum_{(\lambda,\mu)\vdash k} \widetilde{m}_{\lambda,\mu}\chi_{\lambda,\mu}$$

the decomposition of its k-th proper φ-cocharacter. Then $\widetilde{m}_{\lambda,\mu} = 1$, for some pair of partitions $(\lambda, \mu) \vdash k$ if and only if $\dim\,(V^{\lambda,\mu} \cap Id^{\varphi}(\mathcal{V})) = \overline{m}_{\lambda,\mu} - 1$.

For a fixed par of partitions $(\lambda, \mu) \vdash k$ such that $\overline{m}_{\lambda,\mu} \neq 0$ in the decomposition of $\chi(\Gamma_k^{\varphi})$, we define

$$Q^{\lambda,\mu} = \{f_{\sigma,\tau} \,:\, (\sigma, \tau) \vdash k, (\sigma, \tau) \neq (\lambda, \mu)\},$$

and consider

- $W^{\lambda,\mu}$ a subspace de $V^{\lambda,\mu}$ with $\dim W^{\lambda,\mu} = \overline{m}_{\lambda,\mu} - 1$;
- $B^{\lambda,\mu}$ a basis of $W^{\lambda,\mu}$;
- $h \in V^{\lambda,\mu}$ such that $h \notin W^{\lambda,\mu}$.

We define the following T^{φ}-ideals:

$$I_k = \begin{cases} \left\langle \Gamma_{k+1}^{\varphi}, Q^{\lambda,\mu}, B^{\lambda,\mu}, [y_1, y_2]\cdots[y_{k+1}, y_{k+2}]\right\rangle_{T^{\varphi}}, & \text{if } k \text{ is even and } \varphi \text{ is an automorphism} \\ \left\langle \Gamma_{k+1}^{\varphi}, Q^{\lambda,\mu}, B^{\lambda,\mu}\right\rangle_{T^{\varphi}}, & \text{otherwise} \end{cases};$$

$$I_{k-j} = \left\langle I_{k-j+1}, f_{\delta,\xi} \,:\, (\delta, \xi) \vdash (k-j) \text{ such that } h \notin \langle f_{\delta,\xi}, I_{k-j+1}\rangle_{T^{\varphi}}\right\rangle_{T^{\varphi}},$$

for $1 \leq j \leq k-2$.

We have

$$I_k \subseteq I_{k-1} \subseteq \cdots \subseteq I_2.$$

Notice that the T^{φ}-ideal I_2 depends on the choice of the subspace $W^{\lambda,\mu}$ and on the Y-proper highest weight vector h.

We introduce the notation $I_2 = I_h^{\lambda,\mu}$, and we have proved the following result.

Theorem 11. *Let $\mathcal{V}$ be a φ-variety of growth $n^k, k \geq 4$, generated by a unitary algebra. Then $\mathcal{V}$ is minimal if and only if $Id^{\varphi}(\mathcal{V}) = I_h^{\lambda,\mu}$, for some pair of partitions $(\lambda, \mu) \vdash k$.*

References

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