Revisiting Gödel's Koan

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In 1938, Gentzen published a new version of the consistency proof for elementary arithmetic ([3]). In this proof he used for the second time transfinite induction up to the ordinal ϵ_0. In 2015 Dag Prawitz (see [7]) proved for the first time Gentzen's result for a Natural Deduction formalization of Peano's arithmetic and showed in a very interesting way how the fundamental ideas in Gentzen's proof were related to the elimination of (local) detours in Natural Deduction. We can describe the general structure of Gentzen's proof as follows: Consider a formulation of arithmetic in sequent calculus.

1. Define an assignment Ord of ordinals $< \epsilon_0$ to proofs in the system.
2. Define a set of reduction operations OP.
3. Show that if there is a proof Π of the empty sequent in the system, then there is always an operation op $\in$ OP such op[Π] is a proof of the empty sequent and Ord[op[Π] $<$ Ord[Π] .
4. The result immediately follows by transfinite induction up to ϵ_0.

In 1982 (see [6]) it was shown that the reductions used by Gentzen in this new version of the consistency could be used to obtain a cut-elimination proof for Gentzen's LK. The proof was carried out by induction on a natural number assignment to proofs in LK. This assignment produced a better estimation for the length of cut-free proofs in LK. It was shown in 1996 (see [4]) that in the case of the propositional fragment of LK, Gentzen's reductions always yield a smaller natural number and hence that they could be used to obtain

a strong cut-elimination result for this fragment. This natural number assignment, that depends solely on the structure of the proof, obviously provides a bound for the longest reduction sequence. It was suggested in the paper that the same result could be obtained for the propositional fragment of the intuitionistic system FIL, a system where the usual cardinality restriction on the consequent of intuitionistic sequents is replaced by a sort of restriction on dependency relations (see [1]).
In 1968 William Howard proposed an assignment of ordinals $< \epsilon_0$ to terms for primitive recursive functionals of finite type with the property that the reduction of a term always lowers its ordinal measure. It was Howard's assignment that motivated the apearence in 2014 of problem 26 (submitted by Henk Barendregt) in the TLCA list of open problems, the so-called Gödel's Koan.

"Statement: Assign (in an "easy" way) ordinals to terms of the simply typed lambda calculus such that reduction of the term yields a smaller ordinal."

We will examine in this paper several strategies to solve Gödel's Koan and propose an extension of Gentzen's assignment and reductions to the implicational fragment of LJ that, under certain conditions, can be considered an "easy"solution to Gödel's Koan.

References

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