Andrade rheology in time domain, its applications to planetary systems.

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Abstract

We address the problem of dissipation of energy due to tidal deformations. Dissipation of energy by tides depends on the rheological model used to describe the body deformation. It has been argued that among the several visco-elastic models used in geophysics, Kelvin-Voigt, Maxwell, Burger, etc, the most suitable for modeling tidal deformations is that of Andrade. Here we present finite dimensional sets of ordinary differential equations that approximate the Andrade rheology with an arbitrary degree of accuracy. As an application we consider librations and anomalous dissipation rate of one of satellites of Saturn, Enceladus.

Introduction

In our study we address energy dissipation by celestial bodies due to tidal deformations. Dissipation of energy by tides depends on the rheological model used to describe the body deformation. It has been argued that among the several visco-elastic models used in geophysics, Kelvin-Voigt, Maxwell, Burger, etc, the most suitable for modeling tidal deformations is that of Andrade. The challenge of using Andrade rheology is that it is represented by fractional derivatives in the time domain, which are integro-differential operators. For this reason this is for the first time that equations of motion in the time domain that describe simultaneously the rotation the position and tide deformation under Andrade rheology are considered. We present finite dimensional sets of ordinary differential equations that approximate the Andrade rheology with an arbitrary degree of accuracy. To validate the method we analyze the energy dissipation of a celestial body, Enceladus, using the Andrade rheology to describe tidal deformation.

The Enceladus

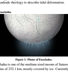

Figure 1: Plums of Enceladus.

Enceladus is one of the medium sized moons of Saturn with a radius of 252.1 km, mostly covered by ice. Currently it is synchronized with its host planet. It is orbiting the planet in a slyghtly eccentric orbit due to it being in 1 : 2 mean motion resonance with another satellite of Saturn, Dione. During the flyby of July 14th, 2005, the Cassini Composite Infrared Spectrometer (CIRS) found plume activity around four large fractures near the south pole of Enceladus. The latest estimate for the energy flux from the fractures is around 10 GW [1]. Here our goal is to obtain dissipation rate and librations by numerical integrations of unaveraged equations of motion. We consider the Andrade rheology for the interior of Enceladus.

Enceladus, Dion and Saturn

To include the influence of Dion on Enceladus-Saturn system we consider Enceladus on a fixed slightly eccentric orbit around Saturn. In the inertial frame the equations of motion are

$$\begin{aligned}
\dot{\omega} &= \frac{3Gm_2}{|q'|^5}[q' \otimes q', \mathbf{b}] \\
\dot{q}' &= v \\
\dot{v} &= GM \\
&\left\{ -\frac{1}{|q'|^3}q' - \frac{15}{2}\frac{I_o}{m_1}\frac{1}{|q'|^7}(q' \cdot \mathbf{b}q')q' + 3\frac{I_o}{m_1}\frac{1}{|q'|^5}\mathbf{b}q' \right\}
\end{aligned} \tag{1}$$

The rheologies

The dissipation equations of motion are the equations for the non-dimensional quadrupole moment. The derivation of the equations for the non-dimensional quadrupole moment is done by means of the **Association Principle** (**AP**) [3]. To use the **AP** we have to present the rheological model by viscoelastic oscillators. The Andrade viscoelastic oscillator is the one in the Figure 2.

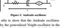

Figure 2: Andrade oscillator.

We were able to show that the Andrade oscillator can be replaced by the generalized Voight oscillator in the Figure 3. It is much more convenient to work with the generalized Voigt model.

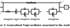

Figure 3: Generalized Voigt oscillator associated to the Andrade model.

For the generalized Voight rheology in the inertial reference frame the equations of motion are

$$\begin{aligned}
\dot{\mathbf{b}} &= [\omega, \mathbf{b}] + \frac{J_2}{1+\gamma J_0}(\dot{\mathbf{f}} - [\omega, \mathbf{f}]) \\
&+ \frac{\sum_{\nu=0}^{n}\eta_\nu^{-1}}{1+\gamma J_0}(\mathbf{f} - \gamma\mathbf{b}) - \frac{\sum_{\nu=1}^{n}\tau_\nu^{-1}\mathbf{b}_\nu}{1+\gamma J_0} \\
\dot{\mathbf{b}}_1 &= [\omega, \mathbf{b}_1] - \tau_1^{-1}\mathbf{b}_1 + \eta_1^{-1}(\mathbf{f} - \gamma\mathbf{b}) \\
\dot{\mathbf{b}}_2 &= [\omega, \mathbf{b}_2] - \tau_2^{-1}\mathbf{b}_2 + \eta_2^{-1}(\mathbf{f} - \gamma\mathbf{b}) \\
&\dots \\
\dot{\mathbf{b}}_n &= [\omega, \mathbf{b}_n] - \tau_n^{-1}\mathbf{b}_n + \eta_n^{-1}(\mathbf{f} - \gamma\mathbf{b}),
\end{aligned} \tag{2}$$

with

$$\mathbf{f} = -\omega^2 + \frac{1}{3}\mathrm{Tr}(\omega^2)\mathbb{I} + \frac{3Gm_2}{|q'|^5}\left(q' \otimes q' - \frac{|q'|^2}{3}\mathbb{I}\right). \tag{3}$$

These equations, together with the ones for the orbit, are the full set of the equations of motion to be integrated in this work.

Expected forced librations

Forced librations may be an important source of energy dissipation in Enceladus (see [2]). Because in our numerical setting we integrate the non-averaged equations of motion, a forced libration naturally appears in the simulations. From the equations of motion we can derive an analytical expression for the libration amplitude

$$\hat{\zeta} = \frac{-ic_1^2[K_2(n) - K_2(0)]}{n^2 + c_1^2[K_2(n) - K_2(0)]}e\,, \tag{4}$$

with $c_1 = 3n$ and K_2 beeing the Love number. We have an analytical expression for the dissipation rate as well

$$\begin{aligned}
\frac{\Delta E}{T} &= \Im[k_2(n)]\frac{(nR)^5}{G}\left(\frac{m_2}{m_1+m_2}\right)^2 \\
&\times \left[\frac{21}{2}e^2 + 12e\Im(\hat{\zeta}) + 6|\hat{\zeta}|^2\right].
\end{aligned} \tag{5}$$

We compare the results from these expressions with ones from the direct integration of equations of motion.

Integration of the equations of motion

After specifying all the initial conditions we can numerically integrate the equations of motion.

Forced librations The amplitude of the forced libration can be recovered once the free libration is damped. The main component of the forced libration is oscillating at the orbital frequency n. Therefore, once the free libration is totally damped, the amplitude is given by

$$\text{Ampl. of Lib.} = \sqrt{\left(\frac{\Omega_1}{n} - 1\right)^2 + \left(\frac{\Omega_3}{n^2}\right)^2}\,.$$

This quantity, obtained by numerical integration of the equations of motion, is plotted in the Figures 4 and 5. The value obtained is around 0.00051 rad. The observed magnitude of libration is 0.0021 rad [4]. We obtain the forced librations for the entire body on the other hand the observed libration is just the one for the mantle, using the two values we estimate the libration amplitude for the core to be 0.6% of that for the mantle.

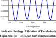

Figure 4: Andrade rheology: Libration of Enceladus in relative normalized spin rate, $(\omega - n)/n$, for four complete orbital periods.

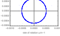

Figure 5: Andrade rheology: Libration of Enceladus in relative normalized spin rate, $(\omega - n)/n$, for four complete orbital periods.

Dissipation rate With the numerical integrations, we have access to the instantaneus dissipation rate within the system. Plotted in Figure 6 in good agreement with the calculation.

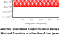

Figure 6: Andrade (generalized Voight) rheology: Dissipation rate (in Watts) of Enceladus as a function of time (years).

Acknowledgment

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