Wave models with time-dependent potential and speed of propagation

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Abstract

We study the long time behavior of energy solutions for a class of wave equation with time-dependent potential and speed of propagation. We introduce a classification of the potential term, which clarifies whether the solution behaves like the solution to the wave equation or Klein-Gordon equation. Moreover, the derived linear estimates are applied to obtain global (in time) small data energy solutions for the Cauchy problem to semilinear Klein-Gordon models with power nonlinearity.

Introdução

Let us consider the Cauchy problem for the wave equation with time-dependent potential and speed of propagation

$$\begin{cases} u_{tt} - a(t)^2\Delta u + m(t)^2 u = 0, & (t,x) \in (0,\infty)\times\mathbb{R}^n, \\ (u(0,x), u_t(0,x)) = (u_0(x), u_1(x)), & x \in \mathbb{R}^n. \end{cases} \tag{1}$$

The Klein-Gordon type energy for the solution to (1) is given by

$$E_{a,m}(t) \sim \|u_t(t,\cdot)\|^2_{L^2} + a(t)^2\|\nabla_x u(t,\cdot)\|^2_{L^2} + m(t)^2\|u(t,\cdot)\|^2_{L^2} \tag{2}$$

One can observe many different effects for the behavior of $E_{a,m}(t)$ as $t\to\infty$ according to the properties of the speed of propagation $a(t)$ and the coefficient $m(t)$ in the potential term.

We first discuss properties of the energy in the case $m(t)\equiv 0$ in (1). If $0 < a_0 \le a(t) \le a_1$ for any $t \ge 0$ with a suitable control of the oscillations it is possible to prove that $E_{a,0}(t)$ has the so-called *generalized energy conservation* property. Bui/Reissig proved energy estimates considering $a(t) \ge a_0 > 0$ an increasing function also satisfying suitable control on the oscillations.

In the case $a(t)\equiv 1$, $E_{1,m}(t)$ is a conserved quantity for the classical Klein-Gordon equation, whereas it is known that the behavior of the potential energy $\|u(t,\cdot)\|_{L^2}$ changes accordingly to the cases $\lim\limits_{t\to\infty} tm(t) = \infty$ or $\lim\limits_{t\to\infty} tm(t) = 0$. To explain this effect, let us consider the energy

$$E_p(u)(t) \doteq \frac{1}{2}\Big(\|u_t(t,\cdot)\|^2_{L^2} + \|\nabla_x u(t,\cdot)\|^2_{L^2} + p(t)^2\|u(t,\cdot)\|^2_{L^2}\Big).$$

Bohme/Reissg studied decreasing coefficients $m = m(t)$ which satisfy among other things $\lim_{t\to\infty} tm(t) = \infty$. In this case the potentials are called *effective*, i.e., the decay of solutions and its derivatives is related to the decay of solutions of the classical Klein-Gordon equation measured in the L^q norm. Under some additional condition on m, was proved that $E_p(u)(t) \le CE_p(u)(0)$, with $p(t)^2 = m(t)$. Bohme/Reissig derived the energy estimate $E_p(u)(t) \le CE_p(u)(0)$, for scale invariant models $m(t) = \frac{\mu}{1+t}$, $\mu > 0$, but now the constant μ has an influence on the function $p(t)$.

In [2, 3] the authors explained qualitative properties of solutions to (1) in the case $a\equiv 1$ and $\lim_{t\to\infty} tm(t) = 0$. Under a suitable control on the oscillations of m, if $(1+t)m(t)^2 \in L^1(\mathbb{R}^+)$, it was proved a scattering result to free wave equation, whereas the potentials are called *non-effective* if $(1+t)m(t)^2 \notin L^1(\mathbb{R}^+)$ and $\limsup_{t\to\infty}(1+t)\int_t^\infty m(s)^2 ds < \frac{1}{4}$. In the case of *non-effective* potentials, the decay of the solutions and its derivatives is related to the decay of solutions to the free wave equation measured in the L^q norm.

Objetivos

We introduced a classification for the potentials in (1) in terms of the time-dependent speed of propagation $a(t)\notin L^1$. In the case of *effective* and *non-effective* potentials we derive sharp energy estimates. As an application to our derived linear estimates, we proved global existence (in time) of small data energy solutions, in the case of effective potentials, for semilinear models with power nonlinearity associated to (1).

Resultados

Let $a \in \mathcal{C}^2[0,\infty)$ be a strictly positive function, such that $a\notin L^1$. We define

$$A(t) \doteq 1 + \int_0^t a(\tau)d\tau, \quad \eta(t) \doteq \frac{a(t)}{A(t)}, \quad m(t) = \mu(t)\eta(t) > 0.$$

Theorem 1. *If $a(t)$ and $\mu(t)$ satisfy suitables oscillations conditions, then*

1. *The potential term $m(t)^2u$ generates* scattering *to the corresponding wave model if $\mu^2\eta \in L^1([0,\infty))$.*
2. *The potential term $m(t)^2u$ represents a* non-effective potential *if $\mu^2\eta\notin L^1$ and*

$$\limsup_{t\to\infty} A(t)\left\{\int_t^\infty \frac{\mu(s)^2 a(s)}{A(s)^2}ds + \frac{a'(t)}{2a(t)^2} - \frac{1}{4}\int_t^\infty \frac{[a'(s)]^2}{a(s)^3}ds\right\} < \frac{1}{4}.$$

3. *The potential term $m(t)^2u$ generates an* effective potential *if $\lim_{t\to\infty}\mu(t) = \infty$.*

Theorem 2. *Suppose that potential term $m(t)^2u$ generates an effective potential. If $\frac{\eta}{\mu} \in L^1[0,\infty)$ and $1 < p \le \frac{n}{[n-2]_+}$ such that $\int_0^t a(s)^{-\frac{1-k}{2}} m(s)^{-\frac{p+k}{2}}\left(\frac{m(s)}{a(s)}\right)^{\frac{n(p-1)}{4}} ds < \infty, k = 0,1,$ then there exists a constant $\epsilon > 0$ such that for all $(u_0,u_1) \in H^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ with $||(u_0,u_1)||_{H^1\cap L^2} \le \epsilon$ there exists a uniquely determined energy solution $u \in C([0,\infty), H^1(\mathbb{R}^n)) \cap C^1([0,\infty), L^2(\mathbb{R}^n))$ to the semilinear model with power nonlinearity associated to* (1) .

Referências

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