Trapped submanifolds contained into a null hypersurface of de Sitter spacetime

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Abstract

We study codimension two trapped submanifolds contained into the future component of the light cone of de Sitter spacetime. For codimension two compact spacelike submanifolds in the light cone we show that they are conformally diffeomorphic to the round sphere. This fact enables us to deduce that the problem of characterizing compact marginally trapped submanifolds into the light cone is equivalent to solving the Yamabe problem on the round sphere, allowing us to obtain our main classification result for such submanifolds.

Introduction

Let $\mathbb{L}^{n+3}$ be the $(n+3)$-dimensional Lorentz-Minkowski space and $\mathbb{S}_1^{n+2}$ the standard model of the de Sitter space defined by

$$\mathbb{S}_1^{n+2} = \{x \in \mathbb{L}^{n+3} : \langle x, x\rangle = 1\}$$

endowed with the induced metric from $\mathbb{L}^{n+3}$.

Consider Σ a codimension two spacelike submanifold of de Sitter spacetime, that is, $\psi : \Sigma \to \mathbb{S}_1^{n+2}$ a smooth immersion which induces a Riemannian metric on Σ.

We will consider on $\mathbb{S}_1^{n+2}$ the time-orientation induced by the globally defined timelike vector field $e_0^* \in \mathfrak{X}(\mathbb{S}_1^{n+2})$ given by

$$e_0^*(x) = e_0 - \langle e_0, x\rangle x = e_0 + x_0 x, x \in \mathbb{S}_1^{n+2},$$

where $e_0 = (1, 0, \ldots, 0)$.

The mean curvature vector field is defined by

$$\mathbf{H} = \frac{1}{n}\mathrm{tr}(\mathrm{II})$$

where II is the second fundamental form of the submanifold. The submanifold Σ is said to be future (past) marginally trapped if $\mathbf{H}$ is null ($\langle \mathbf{H}, \mathbf{H}\rangle = 0$) and future-pointing (past-pointing) on Σ.

Submanifolds into the light cone

The future component of the light cone in $\mathbb{S}_1^{n+2}$ with vertex at $\mathbf{a} \in \mathbb{S}_1^{n+2}$ is the subset

$$\Lambda_{\mathbf{a}}^+ = \{x \in \mathbb{S}_1^{n+2} : \langle \mathbf{a}, x\rangle = 1, x_0 > 0, x \neq \mathbf{a}\}.$$

Let $\psi : \Sigma^n \to \mathbb{S}_1^{n+2}$ contained into $\Lambda_{\mathbf{a}}^+$.

We define the function $u : \Sigma \to (0, +\infty)$ setting $u = -\langle \psi, e_0\rangle > 0$.

Then,

$$\xi = \psi - \mathbf{a} \qquad \text{and} \qquad \eta = -\frac{1 + \|\nabla u\|^2 + u^2}{2u^2}\xi + \frac{1}{u}e_0^{\perp}$$

are two globally normal null vector fields along the submanifold which are future-pointing and satisfy $\langle \xi, \eta\rangle = -1$. And

$$A_\xi = I \qquad \text{and} \qquad A_\eta = -\frac{1 + \|\nabla u\|^2 - u^2}{2u^2}I + \frac{1}{u}\nabla^2 u,$$

where $\nabla^2 u : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ is the Hessian operator of u.

Characterization of compact marginally trapped submanifolds into the light cone

From now on, let us suppose that Σ is contained into Λ_a^+, and without loss of generality, we may assume that the vertex of the light cone is the point $\mathbf{a} = (0, \ldots, 0, 1) \in \mathbb{S}_1^{n+2}$, so that

$$\psi(\Sigma) \subset \Lambda^+ = \{x \in \mathbb{S}_1^{n+2} : x_{n+2} = 1, x_0 > 0\}.$$

Proposition 1. *Assume that Σ is complete and that u satisfies*

$$u(p) \leq C\, r(p) \log(r(p)), \quad r(p) \gg 1$$

where C is a positive constant and r denotes the Riemannian distance function from a fixed origin $o \in \Sigma$. Then Σ is compact and conformally diffeomorphic to the round sphere $\mathbb{S}^n$.

Corollary 2. *Assume that Σ is compact. Then there exists a conformal diffeomorphism $\Psi : (\Sigma^n, \langle,\rangle) \to (\mathbb{S}^n, \langle,\rangle_0)$ such that*

$$\Psi^*(\langle,\rangle_0) = \frac{1}{u^2}\langle,\rangle$$

with $\psi = \psi_f \circ \Psi$ where $f = u \circ \Psi^{-1}$ and $\psi_f : \mathbb{S}^n \to \Lambda^+ \subset \mathbb{S}_1^{n+2}$ is

$$\psi_f(p) = (f(p), f(p)p, 1).$$

$$\begin{tikzcd} \Sigma^n \arrow[r, "u"] \arrow[d, shift right, "\Psi^{-1}"'] & (0,+\infty) \\ \mathbb{S}^n \arrow[u, shift right, "\Psi"'] \arrow[ur, "f"'] & \end{tikzcd} \qquad \begin{tikzcd} \Sigma^n \arrow[r, "\psi"] \arrow[d, shift right, "\Psi^{-1}"'] & \Lambda^+ \subset \mathbb{S}_1^{n+2} \\ \mathbb{S}^n \arrow[u, shift right, "\Psi"'] \arrow[ur, "\psi_f"'] & \end{tikzcd}$$

Theorem 3. *Assume that Σ is compact and marginally trapped. Then there exists a conformal diffeomorphism $\Psi : (\Sigma^n, \langle,\rangle) \to (\mathbb{S}^n, \langle,\rangle_0)$ such that $\psi = \psi_{\mathbf{b}} \circ \Psi$, where $f_{\mathbf{b}} : \mathbb{S}^n \to (0, +\infty)$ is*

$$f_{\mathbf{b}}(p) = \frac{1}{\langle p, \mathbf{b}\rangle_0 + \sqrt{1 + \|\mathbf{b}\|_0^2}}$$

for any $\mathbf{b} \in \mathbb{R}^{n+1}$ and $\psi_{\mathbf{b}} : \mathbb{S}^n \to \Lambda^+ \subset \mathbb{S}_1^{n+2}$ is

$$\psi_{\mathbf{b}}(p) = (f_{\mathbf{b}}(p), f_{\mathbf{b}}(p)p, 1).$$

This work is partially supported by Fundación Séneca project 19901/GERM/15 and research grant 19783/FPI/15 and MINECO/FEDER project MTM2015-65430-P, Spain.