A CUMULATIVE RESIDUAL INACCURACY MEASURE FOR COHERENT SYSTEMS AT COMPONENT LEVEL

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1. Abstract

Inaccuracy and information measure are attracting attention in Reliability Theory. In this paper we give a compensator version of the Kumar and Taneja inaccuracy measure of two nonnegative continuous random variables using a point process martingale approach. Its allows to get this measure between two coherent systems by observing their common component's lifetimes.

1. The cumulative residual inaccuracy measure as a metric

The cumulative residual inaccuracy measure (CRI) of S and T defined by Kumar an Taneja (2015) is

$$\varepsilon(S,T) = -\int_0^\infty \overline{F}(x)\log\overline{G}(x)dx = E[-\log\overline{G}(T)].$$

where $\overline{F} = 1 - F$, $\overline{G} = 1 - G$ are the reliability functions of T and S, respectively, and F, G their distribution functions.
We consider to observe two component lifetimes T and S, which are finite positive random variables defined in a complete probability space $(\Omega, \Im, P)$ through the family of sub σ-algebras $(\Im_t)_{t\geq 0}$ of $\Im$, where

$$\Im_t = \sigma\{1_{\{S>s\}}, 1_{\{T>s\}}, 0 \leq s < t\}$$

satisfies Dellacherie's conditions of right continuity and completeness. We assume that $P(S \neq T) = 1$, that is, the lifetimes can be dependent but simultaneous failures are ruled out. In what follows we assume that S and T are totally inaccessible $\Im_t$-stopping time.

1. The cumulative residual inaccuracy measure as a metric

In relation to $(\Im_t)_{t\geq 0}$ and using Doob-Meyer decomposition, we consider the predictable compensator processes $(A_t)_{t\geq 0}$ and $(B_t)_{t\geq 0}$ such that $1_{\{T\leq t\}} - A_t$ and $1_{\{S\leq t\}} - B_t$ are 0 means $\Im_t$-martingales. From the totally inaccessibility of S and T, A_t and B_t are continuous.
Follows, by the well known equivalence results between the distribution functions and the compensator processes that $P(T \leq t|\Im_t) = -\log\overline{F}(t\wedge T) = A_t$ and $P(S \leq t|\Im_t) = -\log\overline{G}(t\wedge S)$. Identifying $-\log\overline{G}(t) = B_t$ and B_t, in $\{S < t\}$ we have

1. The cumulative residual inaccuracy measure as a metric

$$\varepsilon(S,T) = E[\int_0^T B_s ds] = E[\int_0^T(\int_0^s dB_t)ds] =$$

$$E[\int_0^T(\int_t^T ds)dB_t] = E[\int_0^T(T-t)dB_t].$$

As $\psi(u) = T - u$ is a left continuous function, it is an $\Im_t$-predictable process and

$$M_t = \int_0^t (T-u)d(1_{\{S\leq u\}} - B_u)$$

is a mean 0 $\Im_t$ martingale. Then

$$\varepsilon(S,T) = E[1_{\{S\leq T\}}|T-S|].$$

Also, using the same arguments as above we have

$$\varepsilon(T,S) = E[\int_0^S A_s ds] = E[1_{\{T\leq S\}}(S-T)] = E[1_{\{T\leq S\}}|S-T|].$$

1. The cumulative residual inaccuracy measure as a metric

We consider the following definition which is a symmetric generalization of the Taneja and Kumar inaccuracy measure:

Definition 1.1 If S and T are continuous positive random variables defined in a complete probability space $(\Omega, \Im, P)$, we define the cumulative residual inaccuracy measure as

$$CRI_{S,T} = CRI_{T,S} = E[\int_0^T B_s ds] + E[\int_0^S A_s ds] =$$

$$E[1_{\{S\leq T\}}|T-S|] + E[1_{\{T\leq S\}}|S-T|] = E[|T-S|].$$

$CRI_{T,S}$ can be seen as a dispersion when using a lifetime S asserted by the experimenter information of the true lifetime T. Provide that we identify random variables that are equal almost everywhere, $CRI_{S,T}$ is a metric in the L^1 space of random variables. As a metric it has several properties.

1. The cumulative residual inaccuracy measure as a metric

Example 1.2 Using empirical distribution to approximate $CRI_{S,T}$:
When the experimenter information S, of the true lifetime T is one selected from a set of possible system lifetimes, $S_1, S_2, ..., S_n$, .. which are independent and identically distributed as S, with lifetime G, we can consider the random lifetime defined by

$$Y_n(t) = \frac{1}{n}\sum_{i=1}^n 1_{\{S_i\leq t\}},$$

As

$$E[Y_n(t)] = G(t) \;\; and \;\; Var(Y_n(t)) = \frac{G(t)(1-G(t))}{n}$$

converges to 0 when n converges to ∞, $Y_n(t)$ converges to S, in quadratic mean and therefore converges in distribuition to S. As $f(Y_n) = |Y_n|$ is a bounded continuous function $(Y_n(t) \to^D S)$. Folllows that

$$E[|T - Y_n|] \to E[|T-S|].$$

1. The cumulative residual inaccuracy measure as a metric

We observe that

$$E[\int_t^T B_x dx] = E[|T-S|1_{\{t<S\leq T\}}].$$

and extended the $CRI_{S,T}$ concept to the time varying forms corresponding to residual lifetimes in the set $\{t < S\wedge T\}$.
Definition 1.3 If S and T are continuous positive random variables defined in a complete probability space $(\Omega, \Im, P)$, we define the dynamic cumulative residual inaccuracy measure at time t as

$$DCRI^t_{S,T} = E[\int_t^T B_s ds] + E[\int_t^S A_s ds] = E[1_{\{t<S\wedge T\}}|T-S|].$$

1. The cumulative residual inaccuracy measure as a metric

If T and S are two absolutely continuous lifetimes such that A_t is the $\Im_t$-compensator of $1_{\{T\leq t\}}$ and $B_t = \alpha A_t$, $0 < \alpha \leq 1$, is the $\Im_t$- compensator of $1_{\{S\leq t\}}$ we say thet T and S satisfies the proportional risk hazard process.

Theorem 1.4 The characterization Problem If T and S satisfies the proportional risk hazard process, then the dynamic cumulative residual inaccuracy measure $DRCI^t_{S,T} < \infty$ uniquely determines the distribution function of T.

2. Signature point process

Concerning S and T two coherent systems lifetimes with n common components, stochastically dependent, the martingale approach is convenient and the results can be extended on the complete information level.

In our general setup, the complete information level, we consider the vector $(T_1, ..., T_n)$ of n component lifetimes which are finite and positive random variables defined in a complete probability space $(\Omega, \Im, P)$, with $P(T_i \neq T_j) = 1$, for all $i \neq j, i, j$ in $C = \{1, ..., n\}$, the index set of components. The lifetimes can be dependent but simultaneous failures are ruled out.

2. Signature point process

The mathematical description of our observations, the complete information level, is given by a family of sub σ-algebras of $\Im$, denoted by $(\Im_t)_{t\geq 0}$, where

$$\Im_t = \sigma\{1_{\{T_{(i)}>s\}}, 1 \leq i \leq n, 0 < s < t\},$$

satisfies the Dellacherie conditions of right continuity and completeness .

Theorem 2.1 The representation theorem
Let $T_1, T_2, ..., T_n$ be the component lifetimes of a coherent system with lifetime T under the above conditions and notations. Then,

$$P(T \leq t|\Im_t) = \sum_{k=1}^n 1_{\{T=T_{(k)}\}}1_{\{T_{(k)}\leq t\}}.$$

2. Signature point process

Proof
From the total probability rule we have $P(T \leq t|\Im_t) =$

$$\sum_{k=1}^n P(\{T \leq t\} \cap \{T = T_{(k)}|\Im_t) = \sum_{k=1}^n E[1_{\{T=T_{(k)}\}}1_{\{T_{(k)}\leq t\}}|\Im_t].$$

As T and $T_{(k)}$ are $\Im_t$-stopping time and it is well known that the event $\{T = T_{(k)}\} \in \Im_{T_{(k)}}$, see Dellacherie (1972), where

$$\Im_{T_{(k)}} = \{A \in \Im_\infty : A \cap \{T_{(k)} \leq t\} \in \Im_t, \forall t \geq 0\},$$

and then, $\{T = T_{(k)}\} \cap \{T_{(k)} \leq t\}$ is $\Im_t$-measurable.
Therefore $P(T \leq t|\Im_t) =$

$$\sum_{k=1}^n E[1_{\{T=T_{(k)}\}}1_{\{T_{(k)}\leq t\}}|\Im_t] = \sum_{k=1}^n 1_{\{T=T_{(k)}\}}1_{\{T_{(k)}\leq t\}}.$$

3. The cumulative residual inaccuracy measure for coherent systems

Theorem 3.1
Let $T_1, T_2, ..., T_n$, be the common components lifetimes of two coherent systems with lifetimes T and S under the above conditions and notations. Then the cumulative residual accuracy measure of S and T, on the component level, that is, observing $T_i, 1 \leq i \leq n$ is

$$CRI_{S,T} = \int_0^\infty \{\sum_{k=1}^n P(T_{(k)} > t|T = T_{(k)})P(T = T_{(k)}) +$$

$$\sum_{j=1}^n P(T_{(j)} > t|S = T_{(j)})P(S = T_{(j)}) -$$

$$2\sum_{i=1}^n P(T_{(i)} > t|S\wedge T = T_{(i)})P(S\wedge T = T_{(i)})\}.$$

3. The cumulative residual inaccuracy measure for coherent systems

Example 3.2 To proceed we consider the component lifetimes $T_1, T_2, ..., T_n$, of a coherent system with lifetime T which are subject to failures according to a Weibull process with parameters $\beta = 2$ and θ_1 . The lifetime S asserted by the experimenter follows a Weibull process with parameters $\beta = 2$, θ_2. and $S \wedge T$ follows a Weibull process with parameters $\beta = 2$ and $\frac{\theta_1^2\theta_2^2}{\theta_1{}^2 2\theta_2^2}$.
In practical we consider the ordered lifetimes $T_{(1)}, T_{(2)}, ..., T_{(n)}$ with a conditional reliability function given by

$$\overline{F}(t_i|t_1, t_2, ..., t_{i-1}) = \exp[-(\frac{t_i}{\theta})^\beta + (\frac{t_{i-1}}{\theta})^\beta]$$

for $0 \leq t_{i-1} < t_i$ where t_i are the ordered observations.

3. The cumulative residual inaccuracy measure for coherent systems

The reliability function $\overline{F}(t)$ arises also naturally as the reliability function of upper record values in the sequence of independent non-negative random variables $T_1, T_2, ...$ generated from $F(t)$, the time distribution function to the first event (Arnold, Balakrishnan and Nagaraja, 1992).
Considering $T_1, T_2, ..., T_n$ as record values of independent and identically distributed random variables, the events $\{T = T_{(i)}\}$ and $\{T_{(i)} > t\}$ are independent and therefore $P(T_{(i)} > t|T = T_{(i)}) = P(T_{(i)} > t)$. Then

3. The cumulative residual inaccuracy measure for coherent systems

$$CRI_{S,T} = \int_0^\infty [\sum_{k=1}^n s_k^T \exp[-(\frac{t_k}{\theta_1})^\beta + (\frac{t_{k-1}}{\theta_1})^\beta] +$$

$$\sum_{j=1}^n s_j^S \exp[-(\frac{t_j}{\theta_2})^\beta + (\frac{t_{j-1}}{\theta_2})^\beta] -$$

$$2\sum_{i=1}^n s_i^{S\wedge T}\exp[-(\frac{t_i}{\frac{\theta_1^2\theta_2^2}{\theta_1^2+\theta_2^2}})^\beta + (\frac{t_{i-1}}{\frac{\theta_1^2\theta_2^2}{\theta_1^2+\theta_2^2}})^\beta]]dt$$

where s_k^T,s_j^S and $s_i^{S\wedge T}$ are the components vector of s^T, s^S and $s^{S\wedge T}$ of the coherent systems signatures with lifetimes T, S and $S \wedge T$, respectively.
Reference:

Kumar, V. and Taneja, H.C. (2015) Dynamic cumulative residual and past inaccuracy measures. J. *Stat Theory Appl* **14**: 399 - 412.