Representations of $\mathbb{Z} \wr \mathbb{Z}$

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Abstract

A group G is said to be self-similar if admits a faithful representation on an regular one-rooted m-tree $\mathcal{T}_m$ such that the representation is state-closed and is transitive on the tree's first level. In 2016, Dantas, A. and Sidki, S. [1] showed that $\mathbb{Z} \wr \mathbb{Z}$ cannot be self-similar. Although $\mathbb{Z} \wr \mathbb{Z}$ can not be self-similar, recently we show that the group $\mathbb{Z} \wr \mathbb{Z}$ is state-closed of degree 3 and finity by state.

1. Trees and their automorphisms

Let m be a positive integer and consider the alphabet $Y = \{0, 1, ..., m-1\}$ and $\mathcal{M} = \mathcal{M}(Y)$ the set of all finite sequences from Y. The length of an element $u \in \mathcal{M}$ is denoted by $|u|$.

Definition 1 *The 1-rooted regular m-ary tree $\mathcal{T}_m$ is the graph $(V(\mathcal{T}_m), E(\mathcal{T}_m))$ with $V(\mathcal{T}_m) = \mathcal{M}$ and $(u, v) \in E(\mathcal{T}_m)$ if and only if $v = uy$ for some $y \in Y$, where $u, v \in \mathcal{M}$.*

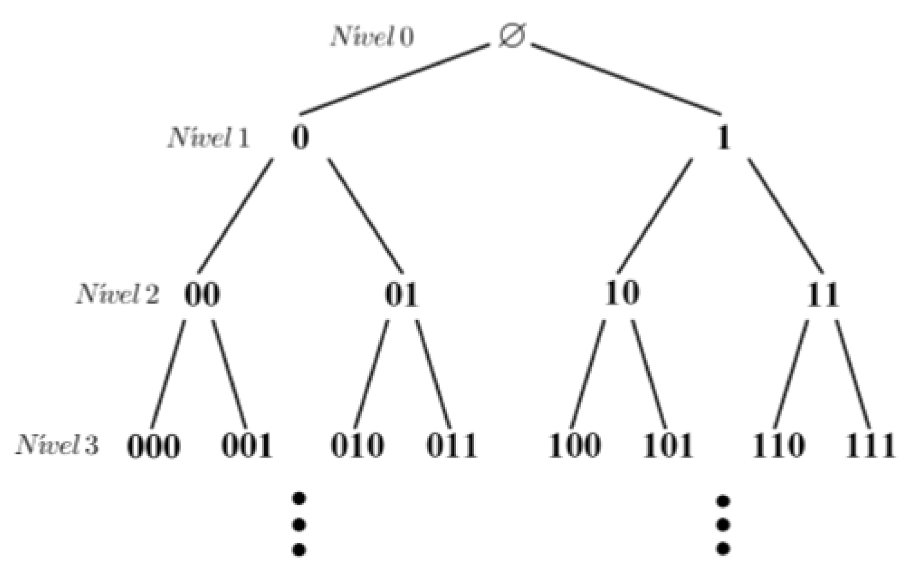

Figure 1: *1-rooted binary tree*

An automorphism α of a 1-rooted regular tree $\mathcal{T}_m$ is a bijection morphism of graphs $\alpha : \mathcal{T}_m \longrightarrow \mathcal{T}_m$, which preserves the length of vertices.

We denote the group of automorphisms of $\mathcal{T}_m$ by $\mathcal{A}_m$.

Example 1 *Given a permutation σ of Y we can extend it (rigidly) to an automorphism $\bar{\sigma}$ of the tree $\mathcal{T}_m$ in the following form:*

$$(\emptyset)\bar{\sigma} = \emptyset,$$

$$(yu)\bar{\sigma} = y^{\sigma}u.$$

An automorphism $\alpha \in \mathcal{A}_m$ induces a permutation $\sigma(\alpha)$ on the set Y. For this we consider the restriction $\alpha|_Y$.

Proposition 1 *The group $\mathcal{A}_m$ satisfies*

$$\mathcal{A}_m = \mathcal{A}_m{}^m \rtimes S_m = \mathcal{A}_m \times ... \times \mathcal{A}_m \rtimes S_m.$$

Thus, we can identify each element $\alpha \in \mathcal{A}_m$ by

$$\alpha = (\alpha_0, ..., \alpha_{m-1})\sigma(\alpha),$$

where $\alpha_y \in \mathcal{A}_m$.

Definition 2 *Given $\alpha = (\alpha_0, ..., \alpha_{m-1})\sigma(\alpha) \in \mathcal{A}_m$, the set*

$$Q(\alpha) = \{\alpha, \alpha_0, ..., \alpha_{m-1}\} \cup Q(\alpha_0) \cup ... \cup Q(\alpha_{m-1})$$

is called of set of states of α.

A subgroup G of $\mathcal{A}_m$ is called state-closed if for all $\alpha = (\alpha_0, ..., \alpha_{m-1})\sigma(\alpha) \in G$ implies that $\alpha_y \in G$ for each $y = 0, ..., m-1$.

If G is an abstract group and there is a homomorphism $\varphi : G \longrightarrow \mathcal{A}_m$ then φ is a representation of G.

The group G is said to be transitive if

$$P(G) = \{\sigma(\alpha) \in S_m : \alpha \in G\}$$

is a transitive subgroup of S_m.

A group G is self-similar provided for some finite positive integer m, the group has a faithful representation on an infinite regular 1-rooted m-tree $\mathcal{T}_m$ such that the representation is state-closed and is transitive on the tree's first level. If a group G does not admit such a representation for any m then we say G is not self-similar.

Theorem 1 *Let G be a group, H a subgroup of G such that $|G : H| = m$, $f \in Hom(H, G)$ and $T = \{t_0, t_1, ..., t_{m-1}\}$ a transversal of H in G. For $g \in G$ let $\sigma(g)$ be the permutation defined by $i^{\sigma(g)} = j$ if and only if $Ht_ig = Ht_j$. Then $\varphi : G \longrightarrow \mathcal{A}_m$ defined by*

$$g^{\varphi} = (t_igt^{-1}_{i^{\sigma(g)}})_{i \in T}\sigma(g)$$

is a representation of G. Furthermore, $\ker(\varphi) = f - core(H)$.

2. Non existence of self-similar representation of $\mathbb{Z} \wr \mathbb{Z}$

Theorem 2 *Let $G = B \wr X$ be a self-similar wreath product of abelian groups. If X is torsion free then B is a torsion group of finite exponent.*

In particular, $\mathbb{Z} \wr \mathbb{Z}$ can not be self-similar.

Steps of Proof

- Let $f : H \longrightarrow G$ be a simple virtual endomorphism with $|G : H| = m$.
- Either $B^m = 1$ or $A_0^f \leqslant A$. In both cases $A \neq A_0$, where $A = B^G$ and $A_0 = A \cap H$.
- If $\exp B$ is infinity, then A^m is a non-trivial normal subgroup of G which is f-invariant.
- Therefore G can not be self-similar.

3. State-closed Representation of $\mathbb{Z} \wr \mathbb{Z}$

Let G be an abstract group and let $H_0, H_1, ..., H_s$ be subgroups of G such that $[G : H_0] = m_0$, $[G : H_1] = m_1$, ..., $[G : H_s] = m_s$ and $m_0 + ... + m_s = m$. Consider $f_0 : H_0 \rightarrow G$, ... , $f_s : H_s \rightarrow G$ virtual endomorphisms. Choice $T_i = \{t_{i1}, ..., t_{im_i}\}$ a transvesal of H_i in G. So we can consider the set

$$\{t_{01}, ..., t_{0m_0}, t_{11}, ..., t_{1m_1}, ..., t_{s1}, ..., t_{sm_s}\}$$

and $\sigma_i(g)$ as the permutation each $g \in G$ induces on T_i. In this conditions, we define $\varphi : G \rightarrow \mathcal{A}_m$ by $g^{\varphi} =$

$$\left([t_{i1}gt^{-1}_{(i1)\sigma_i(g)}]^{f_i\varphi}, ..., [t_{im_i}gt^{-1}_{(im_i)\sigma_i(g)}]^{f_i\varphi}\right)_{0 \leq i \leq s} \sigma_0(g)...\sigma_s(g).$$

Theorem 3 *The function φ is a well defined homomorphism and G^{φ} is state-closed. Moreover*

$$\ker(\varphi) = \langle K \leqslant \bigcap_{i=0}^{s} H_i : K \lhd G, K^{f_i} \leqslant K, \forall i \rangle.$$

A subgroup G of $\mathcal{A}_m$ is state-closed if and only if $\ker(\varphi) = \{1\}$.

Theorem 4 *The group $\mathbb{Z} \wr \mathbb{Z}$ is state-closed of degree 3 and finity by state. Moreover, $\mathbb{Z} \wr \mathbb{Z}$ is isomorphic to*

$$\langle \gamma = (\gamma, \alpha, e), \alpha = (e, e, \alpha)(02) \rangle.$$

Brunner and Sidki [6], proved that if $\alpha = (e, (\alpha, e))\sigma$, $\sigma = (01)$ and H is a subgroup of $\mathcal{A}_2$, then taking

$$\tilde{H} = \{\gamma = ((\gamma, h), e) : h \in H\}$$

we have that

$$G = \langle \tilde{H}, \alpha \rangle$$

satisfies $G/N \simeq H \wr \mathbb{Z}$ where N is a subgroup of G isomorphic with many copies of H'. In particular, if H is abelian, then $G \simeq H \wr \mathbb{Z}$.

Choice $H = \langle \alpha \rangle$, then $\gamma = ((\gamma, \alpha), e)$ and

$$G = \langle \gamma = ((\gamma, \alpha), e), \alpha = (e, (\alpha, e))\sigma \rangle \simeq \mathbb{Z} \wr \mathbb{Z}.$$

G can be embedded in a tree of degree 4 such that G becomes estate-closed

$$G = \langle \gamma = (\gamma, \alpha, e, e), \alpha = (e, e, \alpha, e)(02)(13) \rangle.$$

The embedded can be made on the tree of degree 3, in fact

$$G = \langle \gamma = (\gamma, \alpha, e), \alpha = (e, e, \alpha)(02) \rangle \simeq \mathbb{Z} \wr \mathbb{Z}.$$

For the last isomorphism put $\mathbb{Z} \wr \mathbb{Z} = \langle y \rangle \wr \langle x \rangle$, $H_0 = \langle y \rangle^{\langle x \rangle} \langle x^2 \rangle$ and $H_1 = \mathbb{Z} \wr \mathbb{Z}$. Consider the homomorphisms that extend the maps

$$\begin{aligned} f_0 :\ H_0 &\longrightarrow \mathbb{Z} \wr \mathbb{Z} \\ y^{x^{2n}} &\longmapsto y^{x^n} \\ y^{x^{2n+1}} &\longmapsto e \\ x^2 &\longmapsto x \end{aligned}$$

and

$$\begin{aligned} f_1 : H_1 &\longrightarrow \mathbb{Z} \wr \mathbb{Z} \\ y &\longmapsto x \\ x &\longmapsto e \end{aligned}$$

and so use the representation φ to get $\ker(\varphi) = \{1\}$.

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