Gradient behaviour for large solutions to semilinear elliptic problems

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1. The problem [4], [5], [1], [2]

Given a smooth bounded set $\Omega \subset \mathbb{N}^N$, with $N \geq 1$, $p > 1$ and $f \in W^{1,\infty}(\Omega)$, let us consider the unique large solution $u \in C^2(\Omega)$ of

$$\begin{cases} -\Delta u + |u|^{p-1}u = f & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega. \end{cases} \tag{P}$$

Local $L^\infty(\Omega)$ a priori estimates represent the key idea in proving existence of solution for (P). The achievement of such estimates is strictly related with the following one dimensional problem,

$$-\phi'' + |\phi|^{p-1}\phi = 0, \quad s > 0 \quad \text{and} \quad \lim_{s\to 0^+} \phi(s) = +\infty,$$

whose explicit solution is

$$\phi = \sigma_0 s^{-\alpha} \quad \text{with} \quad \alpha = \frac{2}{p-1}, \quad \sigma_0 = [\alpha(\alpha+1)]^{\frac{1}{p-1}}.$$

Note: Problem (P) admits a solution for a more general nonlinearity $g(u)$ that satisfies the Keller-Osserman growth condition.

$$\exists\, t_0 \;:\; \psi(t) := \int_t^\infty \frac{ds}{\sqrt{2G(s)}} < \infty \;\; t > t_0 \quad \text{where} \quad G'(s) = g(s).$$

Question: Can we describe the explosive behaviour of u near the boundary? Is it affected by the geometry of the domain?

2. Boundary behaviour: what was known

Asymptotic expansion for u [Bandle, Essén, Marcus...]

$$u(x) = \sigma_0 d(x)^{-\alpha} + \sigma_1(x) d(x)^{-\alpha+1} + o(d^{-\alpha+1}(x)) \quad \text{as} \quad d(x) \to 0.$$

where $d(x) \equiv \text{dist}(x, \partial\Omega)$ and

$$\alpha = \frac{2}{p-1}, \quad \sigma_0 = [\alpha(\alpha+1)]^{\frac{1}{p-1}}, \quad \sigma_1(x) = \sigma_0 \frac{\alpha(N-1)H(x)}{2(1+2\alpha)},$$

and $H(x)$ is equal to the curvature of $\partial\Omega$ at point $\bar{x}$, the projection of x on the boundary.

Asymptotic expansion for ∇u [Bandle, Essén, Marcus...]

$$\frac{\partial}{\partial\nu} u(x) = -\alpha\sigma_0 d(x)^{-\alpha-1} + o(d^{-\alpha-1}(x)) \quad \text{as} \quad d(x) \to 0,$$

$$\frac{\partial}{\partial\tau} u(x) = o(d^{-\alpha-1}(x)) \quad \text{as} \quad d(x) \to 0.$$

3. The main result [3]

Theorem 1 Assume Ω of class $C^{[\alpha]+5}$. Then there exists explicit functions $\sigma_k \in C^{[\alpha]+5-k}$ with $k = 0, \cdots, [\alpha]+1$ such that, defining

$$S(x) = \sum_{k=0}^{[\alpha]+1} \sigma_k(x)\, d^{k-\alpha}(x),$$

it results

$$z(x) := u(x) - S(x) \in W^{1,\infty}(\Omega).$$

Moreover it also holds true

$$\forall\, \bar{x} \in \partial\Omega \qquad z(\bar{x}) = 0 \quad \text{and} \quad \lim_{\delta\to 0} \frac{z(\bar{x} - \delta\nu(\bar{x}))}{\delta} = 0\,.$$

Remark 2 It follows that

$$\lim_{d(x)\to 0} \frac{\partial u}{\partial\nu} - \sum_{k=0}^{[\alpha]+1} (\alpha - k)\sigma_k(x)\, d^{k-\alpha-1}(x) + \frac{\partial\sigma_k(x)}{\partial\nu}\, d^{k-\alpha}(x) = 0$$

while

$$\left| \frac{\partial u}{\partial\tau} - \sum_{k=0}^{[\alpha]+1} \frac{\partial\sigma_k(x)}{\partial\tau}\, d^{k-\alpha}(x) \right| \in L^\infty(\Omega)$$

$\forall \tau \in \mathbb{S}^{N-1}$ such that $\tau \cdot \nu = 0$.

4. Theorem's Appendix

Let us provide the explicit expression of the functions σ_k with $k = 0, \cdots, [\alpha]+1$

$$\begin{aligned}
\sigma_0 &:= [\alpha(\alpha+1)]^{\frac{1}{p-1}} \\
\sigma_1(x) &:= -\frac{1}{2}\frac{\alpha\sigma_0}{1+2\alpha}\Delta d(x) = \sigma_0 \frac{\alpha(N-1)H(x)}{2(1+2\alpha)} \\
\sigma_k(x) &:= \frac{(\alpha+1-k)[\sigma_{k-1}(x)\Delta d(x) + 2\nabla\sigma_{k-1}(x)\nabla d(x)] + \Delta\sigma_{k-2}(x)}{(k-\alpha)(k-\alpha-1) - (2+\alpha)(\alpha+1)} \\
&\quad + \frac{\sigma_0^p}{(k-\alpha)(k-\alpha-1) - (2+\alpha)(\alpha+1)} \times \\
&\quad \sum_{j=2}^{k} \left[\binom{p}{j} \sigma_0^{-j} \sum_{i_1+\cdots+i_j = k} \sigma_{i_1}(x) \cdots \sigma_{i_j}(x) \right]
\end{aligned}$$

for $k = 2 \cdots [\alpha]+1$ and $i_1, \cdots, i_j$ positive integers.

5. The proof in sketches

The family of approximating problems. Let us set

$$S_n(x) = \sum_{k=0}^{[\alpha]+1} \sigma_k(x)\, d_n^{k-\alpha}(x), \quad \text{with} \quad d_n(x) = d(x) + \frac{1}{n}, \tag{1}$$

and consider

$$\begin{cases} -\Delta u_n + |u_n|^{p-1}u_n = f, & \text{in } \Omega \\ \frac{\partial u_n}{\partial\nu} = \frac{\partial S_n}{\partial\nu} & \text{on } \partial\Omega. \end{cases}$$

The equation solved by $z_n(x) := u_n(x) - S_n(x)$. It is easy to check that

$$\begin{cases} -\Delta z_n + |z_n + S_n|^{p-1}(z_n + S_n) - |S_n|^{p-1}S_n = \widetilde{f}_n & \text{in } \Omega \\ \dfrac{\partial z_n}{\partial\nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where

$$\widetilde{f}_n = f + \Delta S_n - |S_n|^{p-1}S_n.$$

Steps of the proof.
(i) $u_n \to u$ in $C^2_{loc}(\Omega)$, that implies $z_n \to z$ in $C^2_{loc}(\Omega)$;
(ii) $\exists\, C = C(p, N, \Omega, f)$ such that $||z_n||_{L^\infty(\Omega)} + ||\nabla z_n||_{L^\infty(\Omega)} \leq C$;
(iii) $\forall\, \bar{x} \in \partial\Omega \quad z(\bar{x}) = 0$ and $\lim_{\delta\to 0} \frac{z(\bar{x}-\delta\nu(\bar{x}))}{\delta} = 0$

More details concerning Step (ii) - Bernstein type estimates in a neighborhood of $\partial\Omega$

Inequality satisfied by $|\nabla z_n|^2$. Direct computations shows that $\nabla(|\nabla z_n|^2) = 2D^2 z_n \nabla z_n$

$$\text{and} \quad \Delta(|\nabla z_n|^2) = 2\nabla(\Delta z_n)\nabla z_n + 2|D^2 z_n|^2 \geq 2\nabla(\Delta z_n)\nabla z_n + \frac{2}{N}|\Delta z_n|^2.$$

Taking advantage of (2) and (1) we deduce the existence of δ_0, n_0 and C_1 such that

$$-\Delta(|\nabla z_n|^2) + \gamma\frac{|\nabla z_n|^2}{d_n^2} \leq \frac{C_1}{d_n^2} \quad \text{in} \quad \Omega_0 = \{x \in \Omega \;:\; d(x) < \delta_0\}\,, \quad \forall n > n_0\,. \tag{3}$$

Application of the maximum principle to $w_n := |\nabla z_n|^2 e^{\lambda d_n}$. We can take λ large enough to have

$$\frac{\partial w_n}{\partial\nu} \leq 0 \quad \text{on } \partial\Omega. \tag{4}$$

Moreover, taking in to account (3), it follows that w_n satisfies

$$-\Delta w_n + 2\lambda\nabla w_n\nabla d_n + \frac{\gamma}{2}\frac{w_n}{d_n^2} \leq \frac{C_2}{d_n^2} \quad \text{in} \quad \Omega_0 \quad \text{and} \quad n > n_0. \tag{5}$$

Coupling equation (5) together with the boundary condition (4), we can apply the maximum principle to conclude that

$$\sup_{\Omega_0} w_n \leq C + \max_{\partial\Omega_0\setminus\partial\Omega} w_n \leq C + C_1 \max_{\partial\Omega_0\setminus\partial\Omega} |\nabla z_n|^2.$$

9. References

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