1. The Problem \([4], [5], [1], [2]\)

Given a smooth bounded set \(\Omega \subset \mathbb{R}^N\), with \(N \geq 1\), \(p > 1\) and \(f \in W^{1,p}(\Omega)\), let us consider the unique large solution \(u \in C^2(\Omega)\) of

\[
\begin{cases}
-\Delta u + |u|^{p-2}u = f & \text{in } \Omega, \\
\quad u = +\infty & \text{on } \partial \Omega.
\end{cases}
\]  

(P)

Local \(L^\infty(\Omega)\) a priori estimates represent the key idea in proving existence of solution for \((P)\). The achievement of such estimates is strictly related with the following one dimensional problem,

\[-\phi'' + |\phi|^{p-2}\phi = 0, \quad s > 0 \quad \text{and \; \lim}_{s \to +\infty} \phi(s) = +\infty,\]

whose explicit solution is

\[\phi = \phi_0 s^{-\alpha} \quad \text{with} \quad \alpha = \frac{2}{p-1}, \quad \phi_0 = [(a+1)^p]^{1/p}.\]

Note: Problem \((P)\) admits a solution for a more general nonlinearity \(g(u)\) that satisfies the Keller-Osserman growth condition.

\[\exists t_0: \forall t > t_0 \quad \text{where} \quad G'(s) = g(s).\]

Question: Can we describe the explosive behaviour of \(u\) near the boundary? Is it affected by the geometry of the domain?

4. Theorem’s Appendix

Let us provide the explicit expression of the functions \(\sigma_k\) with \(k = 0, \ldots, |\alpha| + 1\)

\[
\sigma_0 := [(a+1)^p]^{-1},
\sigma_1 := \frac{\alpha a_2}{2(1+2a)} - \frac{\alpha (N-1)H(x)}{2(1+2a)},
\sigma_k := \frac{(a+1-k)|\sigma_{k-1}|\Delta d(x) + 2V_s(-\alpha s)|\nabla d(x)| + \Delta \sigma_{k-1}(x)}{(k-a)(k-a-1)-2(a+1)(a+1) + \sum_{j=2}^{\infty} \sum_{i+i+j=k} \sigma_i \cdot \sigma_j}
\]

for \(k = 2, \ldots, |\alpha| + 1\) and \(i, \ldots, j\) positive integers.

5. The proof in sketches

The family of approximating problems. Let us set

\[S_n(x) := \sum_{k=0}^{|\alpha|+1} \sigma_k d_n^{p-1}(x), \quad \text{with} \quad d_n(x) = d(x) + \frac{1}{n} \quad (1)\]

and consider

\[-\Delta S_n + |S_n|^{p-1}S_n = f, \quad \text{in } \Omega.
\]

The equation solved by \(z_n(x) := u_n(x) - S_n(x)\) is easy to check that

\[-\Delta z_n + |z_n + S_n|^{p-1}(z_n + S_n) - |S_n|^{p-1}S_n = \bar{f}_n \quad \text{in } \Omega.
\]

\[\frac{\partial z_n}{\partial \nu} = 0 \quad \text{on } \partial \Omega,\]

where \(\bar{f}_n = f + \Delta S_n - |S_n|^{p-1}S_n\).

Steps of the proof.

(i) \(u_n \rightarrow u\) in \(C^2_{loc}(\Omega)\), that implies \(z_n \rightarrow z\) in \(C^2_{loc}(\Omega)\);

(ii) \(\exists C = C(p, N, \Omega, f)\) such that \(|z_n|_{L^n(\Omega)} + |\nabla |z_n|_{L^n(\Omega)}| \leq C\);

(iii) \(\forall x \in \partial \Omega: \quad |z| = 0 \quad \text{and} \quad \lim_{x \rightarrow x_0} \frac{|z(x)|}{|x-x_0|} = 0\)

9. References


