Geometric inequalities for critical metrics of the volume functional

Halyson Baltazar, Rondinelle Batista & Ernani Ribeiro Jr.

Universidade Federal do Piauí

rmarcolino@ufpi.edu.br

Instituto de Matemática Pura e Aplicada

Abstract

The goal of this poster is establish sharp estimates to the mean curvature and area of the boundary components of critical metrics of the volume functional on a compact manifold. In addition, localized version estimates to the mean curvature and area of the boundary of critical metrics are also obtained.

Introduction

Let (M^n, g) be a connected Riemannian manifold. We say that g is a V-*static metric* if there is a smooth function f on M^n and a constant κ satisfying the V-static equation

$$\mathfrak{L}_g^*(f) = -(\Delta f)g + Hess\, f - fRic = \kappa g, \qquad (1)$$

where $\mathfrak{L}_g^*$ is the formal L^2-adjoint of the linearization of the scalar curvature operator $\mathfrak{L}_g$. Such a function f is called V-*static potential*.

In the poster we consider *V-static* such that

$$\begin{cases} \mathfrak{L}_g^*(f) = g, & in \quad M \\ f = 0 & on\, \partial M. \end{cases}$$

It is proved in [4] that these metrics arise as critical points of the volume functional on M^n when restricted to the class of metrics g with prescribed constant scalar curvature such that $g_{|\partial M} = h$ for a prescribed Riemannian metric h on the boundary.

Global version of the estimates

Recently, Borghini and Mazzieri [2] used clever methods to provide a new unique- ness result for the de Sitter solution which is essentially based on a new notion of mass established in the realm of static spacetimes with positive cosmological cons- tant that are bounded by Killing horizons. Generically speaking, one of the main techniques used by them to establish the result consists on the analysis via the Maximum Principle of a successful divergence formula.

Firstly in the spirit of [2] and motivated by the historical development on the study of critical metrics of the volume functional, we going to provide a sharp estimate to the mean curvature Hi of the boundary components ∂M_i of a critical metric of the volume functional on an n-dimensional compact manifold. More precisely, we have established the following result.

Theorem 1. *Let (M^n, g, f) be an n-dimensional compact V-static metric with boundary ∂M (possibly disconnected). Then we have:*

$$\min H_i \leq \sqrt{\frac{n(n-1)}{R(f_{\max})^2 + 2nf_{\max}}} \quad on \quad \partial M,$$

where $f_{\max}$ is the maximum value of f. Moreover, the equality holds if and only if M^n is isometric to a geodesic ball in a simply connected space form $\mathbb{R}^n$, $\mathbb{S}^n$ or $\mathbb{H}^n$.

Local version of the estimates

In the sequel we going to establish a localized version of Theorem 1. To this end, we consider $MAX(f)$ to be the set where the maximum of f is achieved, namely,

$$MAX(f) = \{f(p) = M; f(p) = f_{max}\}$$

and let E be a single connected component of $M \setminus MAX(f)$. With these settings we have the following result.

Theorem 2. *Let (M^n, g, f) be an n-dimensional compact V-static metric with boundary ∂M (possibly disconnected), let E be a single connected component of $M \setminus MAX(f)$, and let $\partial E = \partial M \cap E$ be the non-empty and possibly disconnected. Then we have:*

$$\min H \leq \sqrt{\frac{n(n-1)}{R(f_{\max})^2 + 2nf_{\max}}} \quad on\ \partial E.$$

Moreover, the equality holds if and only if M^n is isometric to a geodesic ball in a simply connected space form $\mathbb{R}^n$, $\mathbb{S}^n$ or $\mathbb{H}^n$.

Corollary 1. *Let (M^3, g, f) be a three-dimensional compact V-static metric with boundary (possibly disconnected). Let E be a single connected component of $M \setminus MAX(f)$ and suppose that $\partial E = \cup_{i=1}^k \partial E_i$. Then we have:*

$$\sum_{i=1}^k \frac{1}{H_i}\Big(\frac{R}{6} + \frac{H_i^2}{4}\Big) area(\partial E_i) \leq 2\pi \sum_{i=1}^k \frac{\chi(\partial E_i)}{H_i},$$

Moreover, if the equality holds, then M^3 must be a geodesic ball in $\mathbb{R}^3$, $\mathbb{H}^3$ or $\mathbb{S}^3$.

Finally, in considering that ∂E is connected we immediately obtain the following result which can be compared with Theorem 2 in [1].

Corollary 2. *Let (M^3, g, f) be a three-dimensional compact V-static metric with boundary. Let E be a single component of $M \setminus MAX(f)$ and suppose that ∂E is connected. In addition, for the case of negative scalar we assume that $H^2 > -\frac{2}{3}R$. Then ∂E is a 2-sphere and*

$$area(\partial E) \leq \frac{4\pi}{\left(\frac{R}{6} + \frac{H^2}{4}\right)}.$$

Moreover, the equality holds if and only if M^3 is isometric to a geodesic ball in $\mathbb{R}^3$, $\mathbb{H}^3$ or $\mathbb{S}^3$.

Referências

[1] Batista, R., Diógenes, R., Ranieri, M and Ribeiro, E. Jr.:*Critical metrics of the volume functional on compact three-manifolds with smooth boundary*, J. Geom. Anal. 27 (2017), 1530-1547.

[2] Borghini, S. and Mazzieri, L.: *On the mass of static metrics with positive cosmological constant-I*. Class. Quantum Grav. 35 (2018) 125001.

[3] Corvino, J., Eichmair, M. and Miao, P.:*Deformation of scalar curvature and volume*. Math. Annalen. 357 (2013) 551-584.

[4] Miao, P. and Tam, L.-F.: *On the volume functional of compact manifolds with boundary with constant scalar curvature*. Calc. Var. PDE. 36 (2009) 141-171.

[5] Miao, P. and Tam, L.-F.: *Einstein and conformally flat critical metrics of the volume functional*. Trans. Amer. Math. Soc. 363 (2011) 2907-2937.

Acknowledgements

H. Baltazar was partially supported by FAPEPI Project. Projeto n 007/2018, Brazil. R. Batista was partially supported by research grant from CNPq/Brazil (Grant:310881/2017-0). E. Ribeiro was partially supported by research grants from CNPq/Brazil (Grant: Grant: 305410/2018-0) and PRONEX - FUNCAP / CNPq / Brazil.