On groups with cubic polynomial conditions

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Abstract

Let F_d be the free group of rank d, freely generated by $\{y_1, \cdots, y_d\}$, and let $\mathbb{D}F_d$ be the group ring over an integral domain $\mathbb{D}$. Given a subset E_d of F_d containing the generating set, assign to each s in E_d a monic polynomial $p_s(x) = x^n + c_{s,n-1}x^{n-1} + ... + c_{s,1}x + c_{s,0} \in \mathbb{D}[x]$ and define the quotient ring

$$A_{(d,n,Ed)} = \frac{\mathbb{D}F_d}{\langle p_s(s) | s \in E_d \rangle_{ideal}}$$

When $p_s(s)$ is cubic for all s, we construct a finite set E_d such that $A_{(d,n,E_d)}$ has finite rank over an extension of $\mathbb{D}$ by inverses of some of the coefficients of the polynomials. When the polynomials are all equal to $(x-1)^3$ and $\mathbb{D} = \mathbb{Z}[\frac{1}{6}]$, we construct a finite subset P_d of F_d such that the quotient ring $A_{(d,3,Pd)}$ has finite $\mathbb{D}-rank$ and its augmentation ideal is nilpotent. The set P_2 is $\{y_1, y_2, y_1y_2, y_1^{-1}y_2, y_1^2y_2, y_1y_2^2, [y_1, y_2]\}$ and we prove that $(x-1)^3 = 0$ is satisfied by all elements in the image of F_2 in $A_{(2,3,P_d)}$.

Introduction

The impact of finite order conditions on a group has guided major developments in group theory, so have similar finiteness questions in the theory of algebras [4]. The purpose of this work is to examine finitely generated groups defined by a finite number of algebraic relations of small degree; to wit, polynomials in one variable in degrees 2 and 3. The degree 4 case presents challenging difficulties.

Results

Our first result is a finiteness rank criterion.

Theorem 1. *Let G be a group generated by $\{a_1, ..., a_d\}$. Define the following subsets of G*

$$E_1 = \{a_1\},\ M_1 = \{e, a_1^{\pm 1}\}$$

and inductively for $1 \le s \le d-1$,

$$E_{s+1} = E_s \cup M_s a_{s+1}^{\pm 1},$$
$$M_{s+1} = M_s \cup M_s a_{s+1}^{\pm 1} M_s \cup M_s a_{s+1}^{-1} (M_s \backslash \{e\})\, a_{s+1} M_s,$$

Suppose G is a multiplicative subgroup of a ring R such that each $x \in E_d$ satisfies some cubic polynomial in one variable over the center Z of R. Then the subring of R generated by G is the Z-linear span of M_d.

Let F_d be the free group of rank $d \ge 2$, freely generated by $\{y_1, ..., y_d\}$, $\mathbb{L}$ be an integral domain and $\mathbb{L}F_d$ be the group ring of F_d over $\mathbb{L}$. The augmentation ideal of $\mathbb{L}F_d$ is $\mathbb{L}$- generated by $u(g) = g - 1$ for all $g \in F_d$.

For the sake of completeness, we treat first groups satisfying unipotent quadratic conditions.

Theorem 2. *Define the quotient ring $A_d(2) = \frac{\mathbb{Z}F_d}{\langle (x-1)^2 | x \in S_d \rangle}$ where*

$$S_d = \{y_i\ (1 \le i \le d),\ y_iy_j\ (1 \le i < j \le d)\}$$

and B_d is the augmentation ideal of $A_d(2)$. Then: (i) rank $A_d(2) = 2^d$,(ii) B_d is commutative, nilpotent of degree $d+1$, with torsion subgroup $\mathrm{Tor}(B_d) = B_d^d = \mathbb{Z}.(a_1...a_d - 1)$ *and* $2.B_d^d = 0$;*(iii) G_d is a free d-generated nilpotent group of class 2.*

We pass on to the more general problem of describing finitely generated groups G which satisfy a finite number of cubic unipotent conditions. As expected, the situation here becomes more complex. If we require further that G be a subgroup of a finite dimensional algebra (without characteristic restrictions) and that the unipotent cubic condition holds for all $g \in G$ then by Kolchin's theorem [2] G is nilpotent.

Theorem 3. *Let $\mathbb{D} = \mathbb{Z}\left[\frac{1}{6}\right]$. There exists a finite subset P_d of F_d such that the quotient ring $A_d(3) = \frac{\mathbb{D}F_d}{\langle (x-1)^3 | x \in P_d \rangle}$ has finite $\mathbb{D}$-rank and the augmentation ideal $\omega(A_d(3))$ is nilpotent.*

Remark 1. *We do not know if in our first theorem the condition $(g-1)^3 = 0$ for all $g \in E_d$ yields that the group G is nilpotent.*

The case $d = 2$ of the above theorem is a consequence of a careful study of the following quotient rings

$$A_Q = \frac{\mathbb{Z}F_2}{\left\langle (x-1)^3 \mid x \in Q \right\rangle}, Q = \{y_1, y_2, y_1y_2\};$$
$$A_S = \frac{\mathbb{D}F_2}{\left\langle (x-1)^3 \mid x \in S \right\rangle},\ S = \{y_1, y_2, y_1y_2, y_1^{-1}y_2\};$$
$$A_T = \frac{\mathbb{D}F_2}{\left\langle (x-1)^3 \mid x \in T \right\rangle},\ T = \{y_1, y_2, y_1y_2, y_1^{-1}y_2, y_1^2y_2, y_1y_2^2\};$$
$$A_N = \frac{\mathbb{D}F_2}{\left\langle (x-1)^3 \mid x \in N \right\rangle}, N = \{y_1, y_2, y_1y_2, y_1^{-1}y_2, y_1^2y_2, y_1y_2^2, [y_1, y_2]\}.$$

Theorem 4. *(i) A_Q is freely generated as a $\mathbb{Z}$-module by 1 and monomials in U, V which avoid having subwords from $\left\{(UV)^2, (VU)^2, U^2V^2U^2, V^2U^2V^2\right\}$;*
(ii) A_S has $\mathbb{D}$-rank 23 and $\omega(A_S)$ is nilpotent of degree 7.
(iii) A_T has $\mathbb{D}$-rank 19, $\omega(A_T)^6 = \mathbb{D}.VU^2V^2U$ and $\omega(A_T)^7 = 0$.

The analysis of the above sequence of rings A_Q, A_S, A_T, A_N culminates in a description of A_N. This analysis was done using some routines written in the GAP language, [3], for more details see ricardo.ime.ufg.br.

Theorem 5. *Let $\mathbb{D} = \mathbb{Z}\left[\frac{1}{6}\right]$. Consider the quotient ring $A_N = \frac{\mathbb{D}F_2}{\langle (x-1)^3 | x \in N \rangle}$ where*

$$N = \{y_1, y_2, y_1y_2, y_1^{-1}y_2, y_1^2y_2, y_1y_2^2, [y_1, y_2]\},$$

$\omega(A_N)$ its augmentation ideal and let G be the image of F_2 in A_N. Then
(i) rank $A_N = 18$,
(ii) $\omega(A_N)$ is nilpotent of degree 6,
(iii) G is a free 2-generated nilpotent group of degree 5,
(iv) $(x-1)^3 = 0$ is satisfied by all elements of G.

References

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