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ON A HIGHER DIMENSIONAL VERSION OF THE BENJAMIN-ONO EQUATION

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Abstract

We consider the initial value problem associated to a higher dimensional version of the Benjamin–Ono equation,

$$\begin{cases}\partial_t u - \mathcal{R}_1 \Delta u + u\partial_{x_1} u = 0, \ \ x \in \mathbb{R}^d, \ t \in \mathbb{R}^d, \\ u(x,0) = u_0, \end{cases} \tag{HBO}$$

where $\mathcal{R}_1 = -(-\Delta)^{-1/2}\partial_{x_1}$ denotes the Riesz transform with respect to x_1 and Δ stands for the standard Laplacian operator in the variables $x \in \mathbb{R}^d$. We first establish space-time estimates for the associated linear equation. These estimates enable us to show that the initial value problem for the nonlinear equation is locally well-posed in L^2-Sobolev spaces $H^s(\mathbb{R}^d)$, with $s > 5/3$ if $d = 2$ and $s > d/2 + 1/2$ if $d \geqslant 3$. We also provide ill-posedness results

Introduction.

When $d = 1$ in (HBO) we recover the extensively studied **BO equation**, which servers as a model for long internal gravity waves in deep stratified fluids. The (HBO) equation preserves its physical relevance when $d = 2$, where it describes the dynamics of three–dimensional slightly nonlinear disturbances in boundary-layer shear flows ([1, 4]).

Linear Equation.

Smooth solutions to the linear initial value problem associated to (HBO) can be written as

$$e^{t\mathcal{R}_1\Delta} f(x) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\xi\cdot x} e^{i\xi_1|\xi|t}\widehat{f}(\xi)\, d\xi.$$

Our first result establish Strichartz estimates for the linear operator.

Theorem (Strichartz Estimates)

(i) Let $d \geqslant 3$ and $q < \infty$. Then there is a constant $C \equiv C(d,q,r)$ such that

$$\|e^{t\mathcal{R}_1\Delta} f\|_{L^r_t(\mathbb{R}, L^q_x(\mathbb{R}^d))} \leqslant C\|f\|_{\dot{H}^s(\mathbb{R}^d)}$$

holds for all $f \in \dot{H}^s(\mathbb{R}^d)$ if and only if $\frac{2}{q} + \frac{2}{r} \leqslant 1$ and $s = d(\frac{1}{2} - \frac{1}{q}) - \frac{2}{r}$.

(ii) Let $d = 2$ and $q < \infty$. Then there is a constant $C \equiv C(q,r)$ such that

$$\|e^{t\mathcal{R}_1\Delta} f\|_{L^r_t(\mathbb{R}, L^q_x(\mathbb{R}^2))} \leqslant C\|f\|_{\dot{H}^s(\mathbb{R}^2)}$$

holds for all $f \in \dot{H}^s(\mathbb{R}^2)$ if $\frac{10}{q} + \frac{12}{r} \leqslant 5$ and $s = 1 - \frac{2}{q} - \frac{2}{r}$.

This result is derived from the following proposition after applying a variant of the TT^* argument and the Hardy-Littlewood-Sobolev inequality.

Proposition (Oscillatory Integral)

If $d \geqslant 3$, then there is a constant C, depending only on the Schwartz function ψ supported in $[1/2, 2]$, such that

$$\Big|\int_{\mathbb{R}^d} \psi(|\xi|)\, e^{i[t\xi_1|\xi| + x\cdot\xi]} d\xi\Big| \leqslant C|t|^{-1}, \quad x \in \mathbb{R}^d.$$

Moreover, with $d = 2$,

$$\Big|\int_{\mathbb{R}^2} \psi(|\xi|)\, e^{i[t\xi_1|\xi| + x\cdot\xi]} d\xi\Big| \leqslant C|t|^{-5/6}, \quad x \in \mathbb{R}^2.$$

Moreover, both rates of decay are optimal.

Sketch of the proof.

We write the integral as a two-fold iterated integral

$$I(x,t) := \int_{\mathbb{R}^d} \psi(|\xi|)\, e^{i[t\xi_1|\xi| + x\cdot\xi]} d\xi = \int_0^\infty \rho(r)\Big[\int_{\mathbb{S}^{d-1}} e^{i[tr^2\omega_1 + x\cdot r\omega]} d\sigma(\omega)\Big]\, dr$$
$$= \int_0^\infty \rho(r)\widehat{\sigma}(y(r))dr,$$

where $\rho(r) = \psi(r)r^{d-1}$ and $y(r) = y_{x,t}(r) = (tr^2 + x_1 r, x_2 r, \dots, x_d r)$.

To estimate the right-hand side of the above inequality we use the stationary phase estimate

$$|\widehat{\sigma}(y(r))| \leqslant C_d \min(1, \|y(r)\|^{-(d-1)/2})$$

and the asymptotic expansion

$$\widehat{\sigma}(y(r)) \;=\; c_1\frac{e^{i\|y(r)\|}}{\|y(r)\|^{(d-1)/2}} + c_2\frac{e^{-i\|y(r)\|}}{\|y(r)\|^{(d-1)/2}} + \mathcal{E}_{x,t}(r),$$

valid for large $\|y(r)\|$, where c_1 and c_2 are constants and the uniform estimate $|\mathcal{E}_{x,t}(r)| \leqslant C_d\|y(r)\|^{-(d+1)/2}$ holds. In this manner, the desire conclusion follows from the above estimates and a multi-step argument splitting properly the region of integration.

Nonlinear Equation.

We first notice that the initial value problem associated to (HBO) cannot be solved in $H^s(\mathbb{R}^d)$ by a Picard iterative scheme based on the Duhamel formula.

Theorem (Lack of C^2-regularity)

Let $s \in \mathbb{R}$. Then (HBO) does not admit a solution u such that the flow map $u_0 \mapsto u(t)$ is C^2-differentiable from $H^s(\mathbb{R}^d)$ to $H^s(\mathbb{R}^d)$.

The deduction of local well-posedness for (HBO) is carried out via compactness method following [2, 3] and an approximation procedure based on the Bona-Smith argument.

Theorem (Local well-posedness)

Let $s > s_d$ where $s_d = d/2 + 1/2$ for $d \geqslant 3$ and $s_2 = 5/3$. Then, for any $u_0 \in H^s(\mathbb{R}^d)$, there exist a time $T = T(\|u_0\|_{H^s})$ and a unique solution u to (HBO) that belongs to

$$C\big([0,T]; H^s(\mathbb{R}^d)\big) \cap L^1\big([0,T]; W^{1,\infty}(\mathbb{R}^d)\big).$$

Moreover, the flow map $u_0 \mapsto u(t)$ is continuous from $H^s(\mathbb{R}^d)$ to $H^s(\mathbb{R}^d)$.

As a consequence of our Strichartz estimate, we deduce the following result which is the main ingredient to deduce the above theorem.

Lemma (Refined Strichartz estimate)

Let $s > s_d - 1$ where $s_d = d/2 + 1/2$ for $d \geqslant 3$ and $s_2 = 5/3$. Then

$$\int_0^T \|w(\cdot,t)\|_{L^\infty}\, dt \leqslant C_s T^{1/2}\left(\sup_{t\in[0,T]} \|w(\cdot,t)\|_{H^s} + \int_0^T \|F(\cdot,t)\|_{H^{s-1}}\, dt\right)$$

whenever $T \leqslant 1$ and w is a solution to $\partial_t w - \mathcal{R}_1\Delta w = F$.

References

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