Regularity Theory for the Isaacs equation through approximation methods

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Abstract

The Isaacs equation is an important example of fully nonlinear elliptic equation, appearing in a variety of disciplines. Of particular interest is the fact that such equations are driven by nonconvex operators. Therefore, it falls off the scope of the Evans-Krylov theory and poses additional, delicate, challenges when it comes to its regularity theory. We describe a series of recent results on the regularity theory of the Isaacs equation. These cover estimates in Hölder and Sobolev spaces. We argue through a genuinely geometrical method, by importing information from a related Bellman equation.

Key-words: Isaacs equations; Regularity theory; Estimates in Sobolev and Hölder spaces; Approximation methods

Introduction

This work details previous results obtained in [3], and is part of the author's M.Sc. dissertation, produced under the direction of Professor Edgard Pimentel.

The Isaacs equation has the form

$$\sup_{\alpha\in\mathcal{A}}\inf_{\beta\in\mathcal{B}}\left[-\operatorname{Tr}\left(A_{\alpha,\beta}(x)D^2u\right)\right]=f \quad \text{in } B_1 \tag{1}$$

where $A_{\alpha,\beta}:B_1\times\mathcal{A}\times\mathcal{B}\to\mathbb{R}^{d^2}$ is a (λ,Λ)-elliptic matrix, that means $\lambda\,\mathrm{Id}\le A(x)\le\Lambda\,\mathrm{Id}\quad\forall x\in B_1$.

Definition 1 (Class of viscosity solutions)**.** The Pucci's extremal operators $\mathcal{M}^{\pm}_{\lambda,\Lambda}:\mathcal{S}(d)\to\mathbb{R}$ are defined as follow:

$$\mathcal{M}^{+}(M):=\Lambda\sum_{e_i>0}e_i+\lambda\sum_{e_i<0}e_i$$
$$\mathcal{M}^{-}(M):=\Lambda\sum_{e_i<0}e_i+\lambda\sum_{e_i>0}e_i$$

where e_i are the eigenvalues of M.

Let $f\in\mathcal{C}(B_1)$, a function $u\in\mathcal{C}(B_1)$ is the class of supersolutions $\overline{S}(\lambda,\Lambda,f)$ (resp. subsolutions $\underline{S}$) if

$$\mathcal{M}^{-}(D^2u)\le f \quad \text{in } B_1 \quad (\text{resp. } \mathcal{M}^{+}(D^2u)\ge f)$$

in the viscosity sense. So the class of (λ,Λ)-viscosity solutions is the set

$$S(\lambda,\Lambda,f)=\overline{S}(\lambda,\Lambda,f)\cap\underline{S}(\lambda,\Lambda,f)$$

Regularity theory by approximation methods

Here, we argue through and techniques in the realm of */regularity transmission by approximation methods.* This approach is very much inspired by the idea first introduced in [1]. See also [2].

We can approximate the equation (1) by a Bellman equation of the form

$$\inf_{\beta\in\mathcal{B}}\left[-\operatorname{Tr}\left(A_{\beta}(x)D^2v\right)\right]=0 \quad \text{in } B_{8/9} \tag{2}$$

where $A_\beta:B_1\times\mathcal{B}\to\mathbb{R}^{d^2}$ is also a (λ,Λ)-elliptic matrix, and is under a smallness regime for the quantity $\left|A_{\alpha,\beta}-\overline{A}_\beta\right|$.

If $u\in\mathcal{C}(B_1)$ be a viscosity solution to (1), $\delta>0$ and $q>1$, it's possible to take a Bellman equation's solution $h\in W^{2,q}(B_{7/8})\cap\mathcal{C}(\overline{B}_{8/9})$ in $B_{8/9}$ such that

$$h=u \text{ in } \partial B_{8/9} \quad \text{and} \quad \|u-h\|_{L^\infty(B_{7/9})}<\delta$$

The regularity of h is because (2) is convex with respect to the Hessian, which provides us access to the Evans -Krylov theory that provides solutions with local $C^{2,\gamma}$ and $W^{2,q}$ regularity.

Another useful approximation is through polynomials of the form

$$P_n(x)=a_n+\mathbf{b}_n\cdot x+\frac{1}{2}x^TC_nx$$

that with suitable conditions, satisfies

$$\sup_{x\in B^{\rho^n}}|u(x)-P_n(x)|<\rho^{2(n+\gamma)}$$

and

$$|a_n-a_{n-1}|+\rho^{n-1}\|\mathbf{b}_n-\mathbf{b}_{n-1}\|+\rho^{2(n-1)}\|C_n-C_{n-1}\| \le C\,\rho^{(n-1)(2+\gamma)}$$

where $0<\rho\ll1$ and $\gamma\in[0,1)$

Main Theorems

Estimates in Sobolev Space

Theorem 1. *Let $u\in\mathcal{C}(B_1)$ be a viscosity solution to*

$$\sup_{\alpha\in\mathcal{A}}\inf_{\beta\in\mathcal{B}}\left[-\operatorname{Tr}\left(A_{\alpha,\beta}(x)D^2u\right)-\mathbf{b}_{\alpha,\beta}(x)\cdot Du\right]=f \quad , \text{ in } B_1$$

where $A_{\alpha,\beta}:B_1\times\mathcal{A}\times\mathcal{B}\to\mathbb{R}^{d^2}$ is a (λ,Λ)-elliptic matrix and $\mathbf{b}_{\alpha,\beta}:B_1\times\mathcal{A}\times\mathcal{B}\to\mathbb{R}^d$ is a vector field such that $\mathbf{b}_{\alpha,\beta}\in L^p(B_1)$ uniformly.

For every $\delta>0$ it is possible to choose $\epsilon_1>0$ such that if $\overline{A}_\beta:B_1\times\mathcal{B}\to\mathbb{R}^{d^2}$ satisfies

$$\left|A_{\alpha,\beta}-\overline{A}_\beta\right|<\epsilon_1$$

and $\|f\|_{L^p(\mathcal{O})}<\epsilon_1$, then, $u\in W^{2,p}_{loc}(B_1)$ and

$$\|u\|_{W^{2,p}(B_{1/2})}\le C\left(\|u\|_{L^\infty(B_1)} + \sup_{\alpha\in\mathcal{A}}\sup_{\beta\in\mathcal{B}}\|\mathbf{b}_{\alpha,\beta}\|_{L^p(B_1)}+\|f\|_{L^p(B_1)}\right)$$

where $C>0$ is a universal constant (depends only on d, λ and Λ).

Estimates in $\mathcal{C}^{\text{Log-Lip}}_{loc}(B_1)$

Theorem 2. *Let $u\in\mathcal{C}(B_1)$ be a viscosity solution of*

$$\sup_{\alpha\in\mathcal{A}}\inf_{\beta\in\mathcal{B}}\left[-\operatorname{Tr}\left(A_{\alpha,\beta}(x)D^2u\right)\right]=f \quad , \text{ in } B_1$$

where $A_{\alpha,\beta}:B_1\times\mathcal{A}\times\mathcal{B}\to\mathbb{R}^{d^2}$ is a (λ,Λ)-elliptic matrix.

Suppose that $\overline{A}_\beta:B_1\times\mathcal{B}\to\mathbb{R}^{d^2}$ satisfies that, for every $x_0\in B_1$

$$\sup_{x\in B_{r_0}}\left|A_{\alpha,\beta}(x)-\overline{A}_\beta(x_0)\right|<\epsilon_2$$

uniformly in α and β, and

$$\sup_{r\in(0,r_0]}\fint_{B_r(x_0)}|f(x)-\langle f\rangle|^p\,dx\le\epsilon_2^p$$

for some $\epsilon_2>0$.

If $x_0\in B_{1/2}$, we have

$$\sup_{x\in B_r(x_0)}|u(x)-u(x_0)-Du(x_0)\cdot x|\le C\left(-r^2\ln(r)\right)$$

Estimates in Hölder Space

Theorem 3. *Let $u\in\mathcal{C}(B_1)$ be a viscosity solution of*

$$\sup_{\alpha\in\mathcal{A}}\inf_{\beta\in\mathcal{B}}\left[-\operatorname{Tr}\left(A_{\alpha,\beta}(x)D^2u\right)\right]=f \quad , \text{ in } B_1$$

where $A_{\alpha,\beta}:B_1\times\mathcal{A}\times\mathcal{B}\to\mathbb{R}^{d^2}$ is a (λ,Λ)-elliptic matrix.

Suppose that $\overline{A}_\beta:B_1\times\mathcal{B}\to\mathbb{R}^{d^2}$ satisfies that, for every $x_0\in B_1$

$$\sup_{x\in B_{r_0}}\sup_{\alpha\in\mathcal{A}}\sup_{\beta\in\mathcal{B}}\left|A_{\alpha,\beta}(x)-\overline{A}_\beta(x_0)\right|<\epsilon_3\,r^\gamma$$

and

$$\fint_{B_r}|f(x)|^p\,dx\le\epsilon_3^p\,r^{\gamma p}$$

for some $\epsilon_3>0$.

Then, there exists $\gamma\in(0,1)$ such that u is of class $\mathcal{C}^{2,\gamma}$ at the origin and if $f\equiv0$, we have $u\in\mathcal{C}^{2,\gamma}_{loc}(B_1)$ and there exists a universal constant $C>0$ such that

$$\|u\|_{\mathcal{C}^{2,\gamma}(B_{1/2})}\le\|u\|_{L^\infty(B_1)}$$

References

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Acknowledgements

The author is supported by the Graduate Program in Mathematics at PUC-Rio and CNPq-Brazil.