32° Colóquio Brasileiro de Matemática

IMPA, Rio de Janeiro, July 28 - August 02, 2019

Spectral Theory Approach for a Class of Radial Indefinite Variational Problems

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1. Abstract

Considering the radial nonlinear Schrödinger equation

$$-\Delta u + V(x)u = g(x,u) \text{ in } \mathbb{R}^N,\ N \geq 3 \qquad (P_r)$$

we aim to find a radial nontrivial solution for it, where V changes sign ensuring problem (P_r) is indefinite and g is an asymptotically linear nonlinearity. We work with variational methods associating problem (P_r) to an indefinite functional in order to apply our Abstract Linking Theorem for Cerami sequences in [2] to get a non-trivial critical point for this functional. Our goal is to make use of spectral properties of operator $A := \Delta + V(x)$ restricted to $H^1_{rad}(\mathbb{R}^N)$, the space of radially symmetric functions in $H^1(\mathbb{R}^N)$, for obtaining a linking geometry structure to the problem and by means of special properties of radially symmetric functions get the necessary compactness.

2. Introduction

Since problem (P_r) is radially symmetric, to deal with the Spectral Theory of A it suffices to request informations under an associated operator $\bar{A}$ on the half-line, which is more manageable. Hence we assume that the potential V satisfies:

$(V_1)_r$ $V \in L^\infty(\mathbb{R}^N)$ is a radial sign-changing function, $V(x) = V(|x|) = V(r),\ r \geq 0$;

$(V_2)_r$ Setting $\bar{V}(r) = V(r) + \dfrac{(N-1)(N-3)}{4r^2}$ and $\bar{A} := -\dfrac{d^2}{dr^2} + \bar{V}(r)$, an operator of $L^2(0,\infty)$, $0 \notin \sigma_{ess}(\bar{A})$ and

$$\sup\left[\sigma(\bar{A}) \cap (-\infty, 0)\right] = \sigma^- < 0 < \sigma^+ = \inf\left[\sigma(\bar{A}) \cap (0,+\infty)\right].$$

Moreover, we take the nonlinearity g under the hypotheses below:

(g_1) $g(x,s) \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ is a radial function such that $\lim_{|s|\to 0} \dfrac{g(x,s)}{s} = 0$, uniformly in x and for all $t \in \mathbb{R}$, $G(x,t) = \displaystyle\int_0^t g(x,s)ds \geq 0$;

(g_2) $\lim_{|s|\to+\infty} \dfrac{g(x,s)}{s} = h(x)$, uniformly in x, where $h \in L^\infty(\mathbb{R}^N)$;

(g_3) $a_0 = \inf_{x\in\mathbb{R}^N} h(x) > \sigma^+ = \inf\left[\sigma(A) \cap (0,+\infty)\right]$;

(g_4) $zero$ is not an eigenvalue of $\mathcal{O} := A - \mathcal{H}$, where $\mathcal{H}$ is the operator multiplication by $h(x)$.

(g_5) For $Q(x,s) := \dfrac{1}{2}g(x,s)s - G(x,s) \geq 0$ for all $(x,s) \in \mathbb{R}^N \times \mathbb{R}$ and $\sigma_0 := \min\{\sigma^+, -\sigma^-\}$, there exists $\delta_0 > 0$ such that $\dfrac{g(x,s)}{s} \geq \sigma_0 - \delta_0 \implies Q(x,s) \geq \delta_0$.

In order to get a nontrivial critical point for the indefinite functional associated to (P_r), we apply our following abstract result proved in [2].

Theorem 2.1. ***Linking Theorem for Cerami Sequences:*** *Let E be a real Hilbert space, with inner product $(\cdot,\cdot)$, E_1 a closed subspace of E and $E_2 = E_1^\perp$. Let $I \in C^1(E,\mathbb{R})$ satisfying:*

(I_1) $I(u) = \dfrac{1}{2}\big(Lu,u\big) + B(u)$, *for all $u \in E$, where $u = u_1 + u_2 \in E_1 \oplus E_2$, $Lu = L_1u_1 + L_2u_2$ and $L_i : E_i \to E_i$, $i = 1,2$ is a bounded linear self adjoint mapping.*

(I_2) *B is weakly continuous and uniformly differentiable on bounded subsets of E.*

(I_3) *There exist Hilbert manifolds $S, Q \subset E$, such that Q is bounded and has boundary ∂Q, constants $\alpha > \omega$ and $v \in E_2$ such that*

(i) $S \subset v + E_1$ and $I \geq \alpha$ on S; (ii) $I \leq \omega$ on ∂Q; (iii) S and ∂Q link.

(I_4) *If for a sequence (u_n), $I(u_n)$ is bounded and $(1 + ||u_n||)\,||I'(u_n)|| \to 0$, as $n \to +\infty$, then (u_n) is bounded.*

Then I possesses a critical value $c \geq \alpha$.

3. Variational Setting, Linking Structure and Boundedness

Considering $A := -\Delta + V(x)$ as an operator of $L^2(\mathbb{R}^N)$, since $V \in L^\infty(\mathbb{R}^N)$, A as well as $\bar{A}$ are self-adjoint operators. Due to Hardy's Inequality, operator $\bar{A}$ is treated in $H_0^1(0,\infty)$, which can be written as $H_0^1(0,\infty) = H^- \oplus H^0 \oplus H^+$, with $H^-,\ H^0,\ H^+$ the subspaces of $H_0^1(0,\infty)$ where $\bar{A}$ is respectively negative, null and positive definite. In view of $(V_2)_r$ each $u \in H^+$ satisfies $\sigma^+||u||^2_{L^2(0,\infty)} \leq (\bar{A}u,u)_{L^2(0,\infty)}$. Moreover, given $u \in H_0^1(0,\infty)$ and setting $w := r^{\frac{1-N}{2}}u$, in view of [4] it yields $w \in H^1_{rad}(\mathbb{R}^N)$. In addition, $||w||^2_{L^2(\mathbb{R}^N)} = \omega_N \displaystyle\int_0^\infty |u(x)|^2 dr = \omega_N ||u||^2_{L^2(0,\infty)}$, and $(Aw,w)_{L^2(\mathbb{R}^N)} = \omega_N(\bar{A}u,u)_{L^2(0,\infty)}$, where ω_N is the $(N-1)$-dimensional surface measure of the sphere $S^{N-1} \subset \mathbb{R}^N$. Hence, writing $H^1_{rad}(\mathbb{R}^N) = E = E^- \oplus E^0 \oplus E^+$, with $E^-,\ E^0,\ E^+$ the subspaces where A is respectively negative, null and positive definite, if $w \in E^+$ it satisfies $\sigma^+||w||^2_{L^2(\mathbb{R}^N)} \leq (Aw,w)_{L^2(\mathbb{R}^N)}$. Following the same idea, it yields

$$\sigma^+ = \inf_{w\in E^+} \frac{(Aw,w)_{L^2(\mathbb{R}^N)}}{||w||^2_{L^2(\mathbb{R}^N)}} \quad \text{and} \quad -\sigma^- = \inf_{w\in E^-} \frac{-(Aw,w)_{L^2(\mathbb{R}^N)}}{||w||^2_{L^2(\mathbb{R}^N)}}, \qquad (3.1)$$

which allows us to define an equivalent norm in E given by the expression

$$||u||^2 = ||u||^2_E := ||u^0||^2_{L^2(\mathbb{R}^N)} + \frac{1}{2}(Au^+,u)_{L^2(\mathbb{R}^N)} - \frac{1}{2}(Au^-,u)_{L^2(\mathbb{R}^N)}.$$

From hypothesis $(V_2)_r$ either $0 \notin \sigma(\bar{A})$ or it is an isolated eigenvalue of $\bar{A}$. Since by assumption $0 \notin \sigma_{ess}(\bar{A})$, if $0 \in \sigma(\bar{A})$ it is an eigenvalue of finite multiplicity, hence $\ker(\bar{A})$ is finite dimensional. The same conclusions hold for A, since there exists a correspondence between the eigenfunctions of $\bar{A}$ and the radial eigenfunctions of A. Furthermore, $u_1, u_2 \in H_0^1(0,\infty)$ are orthogonal in $L^2(0,\infty)$ iff $w_1 = r^{\frac{1-N}{2}}u_1$ and $w_2 = r^{\frac{1-N}{2}}u_2$ are orthogonal in $L^2(\mathbb{R}^N)$. Therefore, H^i is infinite dimensional iff E^i is infinite dimensional, for $i = -,\ 0,\ +$.

The functional $I : H^1(\mathbb{R}^N) \to \mathbb{R}$ associated to problem (P_r) is given by

$$I(u) = \Big(||u^+||^2 - ||u^-||^2\Big) - \int_{\mathbb{R}^N} G(x,u)\,dx, \qquad (3.2)$$

which is $C^1(E,\mathbb{R})$ and whose critical points are weak solutions for (P_r) by the Principle of Symmetric Criticality.

Under all the previous assumptions and notations, it is possible to state our main result.

Theorem 3.1. *Suppose $(V_1)_r - (V_2)_r$ and $(g_1) - (g_5)$ hold. Then problem (P_r) possess a radial, nontrivial, weak solution in $H^1(\mathbb{R}^N)$.*

In order to prove Theorem 3.1, it is necessary to show that I satisfies $(I_1) - (I_4)$ in Theorem 2.1 and then obtain a critical point of I, which is a weak solution for (P).

- (I_1) is obtained by writing $\dfrac{1}{2}(Lu,u) := ||u^+||^2 - ||u^-||^2$ and $B(u) := -\displaystyle\int_{\mathbb{R}^N} G(x,u)\,dx$;
- (I_2) is proved indirectly making use of hypotheses $(g_1) - (g_2)$ and the compact embeddings $E \hookrightarrow L^s(\mathbb{R}^N)$ for $2 < s < 2^*$, to get the necessary compactness for $B(u)$;
- (I_3) is for the linking geometry, we choose $Q := \{re + u_2 : r \geq 0, u_2 \in E_2, ||re + u_2|| \leq r_1\}$ and $S = \partial B_\rho \cap E_1$, where $0 < \rho < r_1$ are constants and $e \in E_1$, $||e|| = 1$, must be a suitable vector. In fact, since a_0 in (g_3) is such that $a_0 > \sigma^+$, then for $\varepsilon > 0$ small enough, in view of (3.1) there exists some unitary $e \in E^+$ such that

$$\frac{a_0}{2}||e||^2_{L^2(\mathbb{R}^N)} > \frac{1}{2}(\sigma^+ + \varepsilon)||e||^2_{L^2(\mathbb{R}^N)} \geq \frac{1}{2}(Ae,e)_{L^2(\mathbb{R}^N)} = ||e||^2 = 1, \qquad (3.3)$$

which is chosen for the structure of Q. Since such S and ∂Q "link", the following lemma shows that I satisfies (I_3) $(i) - (ii)$ in Theorem 2.1 for some $\alpha > 0, \omega = 0$, and arbitrary $v \in E_2$.

Lemma 3.2. *Under the hypotheses $(V_1)_r - (V_2)_r$ and $(g_1) - (g_3)$ on I, for Q and S as above and for sufficiently large $r_1 > 0$, I satisfies $I|_S \geq \alpha > 0$ and $I|_{\partial Q} \leq 0$, for some $\alpha > 0$.*

This lemma is the core of our work and the choice of e satisfying (3.3) is essential for the indirect argument used to prove it. For more details confers [3];

- (I_4) consists of bounding the Cerami sequences of I, which is given by the following lemma also proved indirectly.

Lemma 3.3. *Suppose V satisfies $(V_1)_r - (V_2)_r$ and g satisfies $(g_1) - (g_5)$, then I satisfies (I_4).*

Remark 3.4. *It is worth to highlight the special decay of radial functions in $H^1(\mathbb{R}^N)$, which plays a fundamental role in the proofs of Lemmas 3.2 and 3.3, as well as, the fact that $0 \notin \sigma_{ess}(A)$, which ensures that $\ker(A)$ is finite dimensional enabling us to arrive at necessary contradictions. Furthermore, note that hypotheses $(g_4) - (g_5)$ were assumed only for proving Lemma 3.3, then they can be replaced by any assumption which ensures the boundedness of Cerami sequences.*

4. A Nontrivial Critical Point of I

Finally, it is possible to prove our main result.

Proof of Theorem 3.1. Provided that I satisfies all assumptions $(I_1) - (I_4)$ in Theorem 2.1, it ensures a critical point $u \in E$ of I, with $I(u) = c \geq \alpha > 0$, hence u is a non-trivial critical point of $I : E \to \mathbb{R}$. It implies that $I'(u)v = 0$, for all $v \in H^1_{rad}(\mathbb{R}^N)$. Nevertheless, the Principle of Symmetric Criticality implies that $I'(u)v = 0$ for all $v \in H^1(\mathbb{R}^N)$, namely, u is a critical point of I as a functional defined on the whole $H^1(\mathbb{R}^N)$. Since $I \in C^1(H^1(\mathbb{R}^N),\mathbb{R})$, it yields that u is a nontrivial weak solution of (P_r). In addition, since $u \in E$, it is a radial weak solution. □

5. References

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