Commutative Moufang loops and cubic hypersurface

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Overview

Let $V(k)$ be a cubic hypersurface over a field k. In the book [Yu.Manin, Cubic forms, Amsterdam, 1982] the author proved that on $V(k)$ there exists an universal equivalence $\sim$ such that $M_V(k) = V(k)/\sim$ has structure of quasigroup $\circ$, $x \circ y \in M_V(k)$, if $x, y \in M_V(k)$ Moreover, with new multiplication $x \cdot y = c \circ (x \circ y)$, the set $M_V(k)$ is a commutative loop. The connection of this structure of quasigroup on $M_V(k)$ with $V(k)$ is the follows: if $x \neq y \in V(k)$ and a line $l(x, y)$ intersects $V(k)$ in the point $z \in V(k)$, then $z \sim x \circ y$. Yu.Manin proved that the commutative loop $L = (M_V(k), \cdot)$ has the following decomposition $L = L_2 \times L_3$, where L_2 is an elementary abelian $2-$group and L_3 is a commutative Moufang loop of exponent 3(CML_3). In the same book Yu.Manin asked: (1) there exists cubic hypersurface $V(k)$ for some field k such that $(M_V(k), \cdot)$ is not an abelian group? (2) Let F_n be a free CML_3 with n generators. What is the order of F_n? Those questions are still open. In this work we give an answer on the second question for all $n < 9$.

Introduction

A set L with a binary operation $L \times L \to L : (x, y) \mapsto x \cdot y$ is called a CML_3, if for given $a, b, c \in L$ we have $a^2 \cdot (b \cdot c) = (a \cdot b) \cdot (a \cdot c)$. and $a^3 = e, e \cdot a = a$.

Results

It is not difficult to prove that $|F_n| = 3^{\delta_n}$ and $\delta_3 = 4$. The loop F_3 has the following structure. $F_3 = \mathrm{F}_3^4$ and

$$(x_1, ..., x_4).(y_1, ..., y_4) = (x_1 + y_1, ..., x_3 + y_3, x_4 + y_4 + f), \quad (1)$$

where $f = f(x_1, ...x_3, y_1, ..., y_3) = x_1x_3y_2 + x_2y_1y_3 - x_1y_2y_3 - x_2x_3y_1$.

For $n > 3$ we have the following results, obtained during 40 years by many authors (R.Bruck, J.Smith, A.Grishkov, I.Shestakov, A.Zavarnitsine and others):

$\delta_3 = 4$, $\delta_4 = 12$, $\delta_5 = 49$, $\delta_6 = 220$, $\delta_7 = 1014$.

Those results are founded on the following list of known identities of $CML_3 : ((x, y, z), y, z) = (((x, y, z), z, t), z, v) = 1$.

$$\begin{array}{l}((((a,x,y),z,b),t,c),b,c)((((a,x,z),y,b),t,c),b,c)\\((((a,x,t),y,b),z,c),b,c)^{-1}((((a,x,b),y,z),t,c),b,c)\\((((a,x,c),y,z),t,b),b,c)((((a,x,b),y,c),z,t),b,c) = e.\end{array} \quad (2)$$

The last identity was proved recently in our paper with A.Grishkov and A.Zavarnitsive (not published yet) using computational calculation and new approach. We proved that for F_n there exists multiplication formula given by some special polynomials. The Conjecture that this identity is valid in CML_3-loops was formulated in the paper [Grishkov, A., Shestakov, I. *Commutative Moufang loops and alternative algebras*. Journal of Algebra, v. 333, p. 1-13, 2011.]

Theorem 1. *Let M_8 be the commutative Moufang loop of exponent 3 with 8 generators.*

Then $|M_8| = 3^{4688}$, if in M_8 we have the following identities

$$\begin{array}{l}(((((v,x,a),y,b),z,t),a,c),b,c)\\(((((v,x,y),z,a),t,b),a,c),b,c)^{-1}\\(((((v,x,z),y,a),t,b),a,c),b,c)\\(((((v,x,t),y,a),z,b),a,c),b,c)\\(((((v,x,a),y,z),t,b),a,c),b,c)\\(((((v,x,b),y,z),t,a),a,c),b,c) = e.\end{array} \quad (3)$$

$$\begin{array}{l}(((((v,x,a),y,b),z,c),t,a),b,c)^{-1}\\(((((v,x,z),y,a),t,b),a,c),b,c)^{-1}\\(((((v,x,t),y,a),z,b),a,c),b,c)\\(((((v,x,b),y,z),t,a),a,c),b,c)\\(((((v,x,a),y,z),t,b),a,c),b,c)\\(((((v,x,c),y,z),t,a),a,b),b,c) = e.\end{array} \quad (4)$$

If the identities (3) and (4) are not hold in M_8 then $|M_8| = 3^{4800}$.

If only one of the identities (3) or (4) is not hold in M_8 then $|M_8| = 3^{4744}$.

Reduction of a calculation in F_n to the calculation in some associative algebra.

Let $F_n^1 = F_n$, $F_n^2 = (F_n, F_n, F_n)$, $F_n^{k+1} = (F_n^k, F_n, F_n)$. We consider $L(F_n) = \sum_{k=1}^{n-1} \oplus F_n^k / F_n^{k+1}$.

Then $L(F_n)$ is an algebra with $3-$linear operation $[a_i, a_j, a_k] = (a_i, a_j, a_k) \pmod{F_n^{i+j+k+1}}$, where $a_s \in F_n^s$. Every F_3-space F_n^k/F_n^{k+1} has a basis $v_1, ..., v_{f(k)}$. It is clear that the function $f(k)$ depends on n too. For simplify notations we denote x_{i1} by i. Using the following identity of CML_3-loops $((x, y, z), a, b) = ((x, a, b), y, z)(x, (y, a, b), z)(x, y, (z, a, b))$ we can suppose that v_i may be written in the form of a simple associator $v = v_i = [...[i_0, i_1, i_2], i_3, i_4], ...], i_{2k-3}, i_{2k-2}]$, where i_0 is the minimal number such that $i_0 \neq i_s$ for all $s > 0$. By definition

a) $supp(v) = \{i_0, ..., i_{2k-2}\}$,

b) $type(v) = (a_0, a_1, ..., a_t)$, where $a_r = |\{s | i_s \ appears \ in \ supp(v) \ e$

It is clear that $\sum_{i=0}^t ia_i = 2k - 1$. It is easy to prove that for any two simple associators $v, w \in L(F_n)$ there exists an automorphism ϕ of F_n such that $v^\phi = w$ (here we identify ϕ and induced automorphism of $L(F_n)$) if $type(v) = type(w)$.

By definition W_k is a set of all types of non-zero simple associators from F_n^k/F_n^{k+1} and $W = W(n) = \cup_{i=1}^{2k-4} W_i$. We denote by $g(a)$ for $a \in W_k$ the number of basic elements v_i such that $type(v_i) = a$. If we know the set $W(n)$ and corresponding function $g(a)$ it is easy to calculate $\delta_n = log_3|F_n|$. We define for $a = (a_0, ..., a_s)$, $|a| = \sum_{i=0}^s a_s$, $|a|_i = \sum_{j=i+1}^s a$, $C_a^n = C_n^{|a|} C_{|a|}^{a_0} C_{|a|_0}^{a_1} ... C_{|a|_{s-1}}^{a_s}$.

Then $\delta_n = \sum_{a \in W(n)} C_a^n g(a)$.

Since $W(n-1) \subset W(n)$, if we describe $W(n)$, we have description of all $W(m), m < n$. Moreover, for $m < n$ we have

$W(m) = \{a = (a_0, ..., a_s) \in W(n) | \sum_{i=0}^s a_i \leq m\}$.

Main example.

n $= 8$. $W(8) = \{(1,0), (3,0), (5,0), (3,1), (3,2), (3,3), (5,1)\} \cup \{(7,0), (5,2), (6,0,1)(3,4)\} \cup \{(7,1), (7,0,0,1), (5,3), (3,5)\}$,

$g(1,0) = g(3,0) = g(3,1) = 1, g(3,2) = 1, g(5,0) = 4, g(3,3) = 1, g(5,1) = 5, \ g(7,0) = 20, g(5,2) = 6, g(3,4) = g(6,0,1) = 1, \ g(7,1) = 29, g(7,0,0,1) = 1, 6 \leq g(5,3) \leq 8, g(3,5) = 1,$

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