Fickett's question and Prikry's conjecture

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Abstract

Fred Galvin, Jan Mycielski and Robert M. Solovay showed that, the strong measure zero sets (in a σ-totally bounded metric space) can be characterized by the nonexistence of a winning strategy in the Mycielski-Solovay games. The main result of this characterization is a theorem on Cartesian products which is used to answer the Fickett's question and give a proof of the Prikry's conjecture.

Basic definitions

Let X be a metric space,

- X has *strong measure zero* if, for every sequence $(\varepsilon_n)_{n\in\mathbb{N}} \subset \mathbb{R}_{>0}$, there is a partition $X = \bigcup_{n\in\mathbb{N}} X_n$ with $diam(X_n) \leq \varepsilon_n$ for each n.
- X is *totally bounded* if, for every $\varepsilon \in \mathbb{R}_{>0}$, there is a finite partition $X = \bigcup_{n=1}^{m} X_n$ with $diam(X_n) \leq \varepsilon$ for each $n \leq m$.
- X is *σ-totally bounded* if there is a partition $X = \bigcup_{n\in\mathbb{N}} X_n$ such that each X_n is totally bounded.

Mycielski-Solovay games

The Mycielski-Solovay game $G(X)$

Given a metric space X, we define a game $G(X)$ between two players, White and Black. At move n, first White chooses a real number $\varepsilon_n > 0$, and then Black chooses a set $B_n \subseteq X$ with $diam(B_n) \leq \varepsilon_n$. Black wins a play $(\varepsilon_1, B_1, \varepsilon_2, B_2, \ldots)$ of this game if $\bigcup_{n\in\mathbb{N}} B_n = X$, otherwise White wins. We say that the game $G(X)$ is a win for White (Black) if White (Black) has a winning strategy.

Theorem 1 For any metric space X, the game $G(X)$ is a win for Black if and only if X is countable.

The Mycielski-Solovay game $\hat{G}(X, \mathcal{F})$

Let X be a σ-totally bounded metric space, and let $\mathcal{F} = (F_n)_{n\in\mathbb{N}}$ be a sequence of totally bounded subsets of X such that $\bigcup_{n\in\mathbb{N}} F_n = X$ and $F_n \subseteq F_{n+1}$ for each n. We define the game $\hat{G}(X, \mathcal{F})$. At move n, first White chooses a real number $\varepsilon_n > 0$, and then Black chooses a set $B_n \subseteq F_n$ with $diam(B_n) \leq \varepsilon_n$. Black wins if $\limsup_n B_n = X$, otherwise White wins.

Lemma 1 Given a totally bounded metric space F and a number $\delta > 0$, we can find a nonempty finite collection $\mathcal{B}$ of subsets of F, each of diameter at most δ, such that every subset of F of diameter at most $\frac{1}{3}\delta$ is contained in some member of $\mathcal{B}$.

Theorem 2 If X is a σ-totally bounded metric space, and if $\mathcal{F} = (F_n)_{n\in\mathbb{N}}$ is a sequence of totally bounded subsets of X such that $\bigcup_{n\in\mathbb{N}} F_n = X$ and $F_n \subseteq F_{n+1}$ for each n, then the following statements are equivalent:

1. X does not have strong measure zero;
2. $G(X)$ is a win for White;
3. $\hat{G}(X, \mathcal{F})$ is a win for White.

Galvin-Mycielski-Solovay's theorem about Cartesian products

Vertically dense sets

For metric spaces X and Y, a set $A \subseteq X \times Y$ is *vertically dense* if, for each $x \in X$, the set $\{y \in Y : (x, y) \in A\}$ is dense in Y.

Lemma 2 Let X, Y be metric spaces. Given

$a)$ a compact set $K \subseteq X$,
$b)$ a nonempty open set $W \subseteq Y$, and
$c)$ a vertically dense open set $A \subseteq X \times Y$,

we can find a number $\varepsilon > 0$ such that, for any set $B \subseteq K$ with $diam(B) \leq \varepsilon$, there is a nonempty open set $V \subseteq W$ with $B \times V \subseteq A$.

Theorem 3 If

$a)$ X is a σ-compact metric space,
$b)$ Y is a complete metric space with no isolated points,
$c)$ $Z \subseteq X$ is a strong measure zero set,
$d)$ $A \subseteq X \times Y$ is a vertically dense G_δ set,
$e)$ $U \subseteq Y$ is a nonempty open set, and
$f)$ $D \subseteq Y$ is a dense G_δ set,

then there is a nonempty perfect set $P \subseteq U \cap D$ such that $Z \times P \subseteq A$.

Fickett's question for dense G_δ sets

James Fickett asked whether there is a characterization of the sets X of real numbers such that every dense open set, or every dense G_δ set, contains a homothetic copy of X. The next theorem shows that the answer to Fickett's question for dense G_δ sets is that X has strong measure zero.

Theorem 4 For any set $Z \subseteq \mathbb{R}$ the following statements are equivalent:

1. Z has strong measure zero;
2. there is a number $k \in \mathbb{N}$ such that Z can be covered by k translates of every dense open subset of $\mathbb{R}$;
3. for every dense G_δ set $D \subseteq \mathbb{R}$, there are countable sets $A, B \subseteq \mathbb{R}$ such that $Z \subseteq AD + B$;
4. for any dense G_δ set $D \subseteq \mathbb{R}$ and any nonempty open set $U \subseteq \mathbb{R}$, there is a nonempty perfect set $P \subseteq U \cap D$ such that $Z + P \subseteq D$.

Prikry's conjecture

Karel Prikry observed that a set X of real numbers has strong measure zero if every dense open subset of the real line contains a translate of X; and he conjectured that, conversely, every dense open subset and even every dense G_δ subset of the real line contains a translate of every strong measure zero set. The next corollary shows the Prikry's conjecture.

Corollary 1 For any set $X \subseteq \mathbb{R}$, the following statements are equivalent:

1. X has strong measure zero;
2. every dense open set contains a translate of X;
3. every dense G_δ set contains a translate of X.

Fickett's question for dense open sets

The next theorem answers the Fickett's question for dense open sets.

Theorem 5 For every set $X \subseteq \mathbb{R}$, the following statements are equivalent:

1. for every dense open set $D \subseteq \mathbb{R}$ there exist $a, b \in \mathbb{R}, a > 0$, with $aX + b \subseteq D$;
2. X is the union of a bounded set and a strong measure zero set.

References

[1] F. Galvin, J. Mycielski and R. Solovay. *Strong measure zero and infinite games*. Archive for Mathematical Logic, (2017) 56: 725-732, May 2017.

[2] S. Willard. *General Topology*. Addison-Wesley Publishing Company, 1970.