Abstract

Fred Galvin, Jan Mycielski and Robert M. Solovay showed that, the strong measure zero sets (in a $\sigma$-totally bounded metric space) can be characterized by the nonexistence of a winning strategy in the Mycielski-Solovay games. The main result of this characterization is a theorem on Cartesian products which is used to answer the Fickett’s question and give a proof of the Prikry’s conjecture.

Basic definitions

Let $X$ be a metric space,

- $X$ has strong measure zero if, for every sequence $(\varepsilon_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}_{>0}$, there is a partition $X = \bigcup_{n \in \mathbb{N}} X_n$ with $\operatorname{diam}(X_n) \leq \varepsilon_n$ for each $n$.
- $X$ is totally bounded if, for every $\varepsilon \in \mathbb{R}_{>0}$, there is a finite partition $X = \bigcup_{n=1}^{m} X_n$ with $\operatorname{diam}(X_n) \leq \varepsilon$ for each $n \leq m$.
- $X$ is $\sigma$-totally bounded if there is a partition $X = \bigcup_{n \in \mathbb{N}} X_n$ such that each $X_n$ is totally bounded.

Mycielski-Solovay games

The Mycielski-Solovay game $G(X)$

Given a metric space $X$, we define a game $G(X)$ between two players, White and Black. At move $n$, first White chooses a real number $\varepsilon_n > 0$, and then Black chooses a set $B_n \subseteq X$ with $\operatorname{diam}(B_n) \leq \varepsilon_n$. Black wins a play $(\varepsilon_1, B_1, \varepsilon_2, B_2, \ldots)$ if $\bigcup_{n \in \mathbb{N}} B_n = X$, otherwise White wins. We say that the game $G(X)$ is a win for White (Black) if White (Black) has a winning strategy.

Theorem 1

For any metric space $X$, the game $G(X)$ is a win for Black if and only if $X$ is countable.

The Mycielski-Solovay game $\tilde{G}(X, \mathcal{F})$

Let $X$ be a $\sigma$-totally bounded metric space, and let $\mathcal{F} = (F_n)_{n \in \mathbb{N}}$ be a sequence of totally bounded subsets of $X$ such that $\bigcup_{n \in \mathbb{N}} F_n = X$ and $F_n \subseteq F_{n+1}$ for each $n$. We define the game $\tilde{G}(X, \mathcal{F})$. At move $n$, first White chooses a real number $\varepsilon_n > 0$, and then Black chooses a set $B_n \subseteq F_n$ with $\operatorname{diam}(B_n) \leq \varepsilon_n$. Black wins if $\limsup_n B_n = X$, otherwise White wins.

Lemma 1

Given a totally bounded metric space $F$ and a number $\delta > 0$, we can find a nonempty finite collection $\mathcal{B}$ of subsets of $F$, each of diameter at most $\delta$, such that every subset of $F$ of diameter at most $\frac{1}{2}\delta$ is contained in some member of $\mathcal{B}$.

Theorem 2

If $X$ is a $\sigma$-totally bounded metric space, and if $\mathcal{F} = (F_n)_{n \in \mathbb{N}}$ is a sequence of totally bounded subsets of $X$ such that $\bigcup_{n \in \mathbb{N}} F_n = X$ and $F_n \subseteq F_{n+1}$ for each $n$, then the following statements are equivalent:

1. $X$ does not have strong measure zero;
2. $G(X)$ is a win for White;
3. $\tilde{G}(X, \mathcal{F})$ is a win for White.

Galvin-Mycielski-Solovay’s theorem about Cartesian products

Vertically dense sets

For metric spaces $X$ and $Y$, a set $A \subseteq X \times Y$ is vertically dense if, for each $x \in X$, the set $\{y \in Y : (x, y) \in A\}$ is dense in $Y$.

Lemma 2

Let $X, Y$ be metric spaces. Given

a) a compact set $K \subseteq X$,

b) a nonempty open set $W \subseteq Y$, and

c) a vertically dense open set $A \subseteq X \times Y$,

we can find a number $\varepsilon > 0$ such that, for any set $B \subseteq K$ with $\operatorname{diam}(B) \leq \varepsilon$, there is a nonempty open set $V \subseteq W$ with $B \times V \subseteq A$.

Theorem 3

If

a) $X$ is a $\sigma$-compact metric space,

b) $Y$ is a complete metric space with no isolated points,

c) $Z \subseteq X$ is a strong measure zero set,

d) $A \subseteq X \times Y$ is a vertically dense $G_4$ set,

e) $U \subseteq Y$ is a nonempty open set, and

f) $D \subseteq Y$ is a dense $G_4$ set,

then there is a nonempty perfect set $P \subseteq U \cap D$ such that $Z \times P \subseteq A$.

Fickett’s question for dense $G_4$ sets

James Fickett asked whether there is a characterization of the sets $X$ of real numbers such that every dense open set, or every dense $G_4$ set, contains a homothetic copy of $X$. The next theorem shows that the answer to Fickett’s question for dense $G_4$ sets is that $X$ has strong measure zero.

Theorem 4

For any set $Z \subseteq \mathbb{R}$ the following statements are equivalent:

1. $Z$ has strong measure zero;

2. there is a number $k \in \mathbb{N}$ such that $Z$ can be covered by $k$ translates of every dense open subset of $\mathbb{R}$;

3. for every dense $G_4$ set $D \subseteq \mathbb{R}$, there are countable sets $A, B \subseteq \mathbb{R}$ such that $Z \subseteq A \mathbb{D} + B$;

4. for any dense $G_4$ set $D \subseteq \mathbb{R}$ and any nonempty open set $U \subseteq \mathbb{R}$, there is a nonempty perfect set $P \subseteq U \cap D$ such that $Z \times P \subseteq D$.

Prikry’s conjecture

Karel Prikry observed that a set $X$ of real numbers has strong measure zero if every dense open subset of the real line contains a translate of $X$; and he conjectured that, conversely, every dense open subset and even every dense $G_4$ subset of the real line contains a translate of every strong measure zero set. The next corollary shows the Prikry’s conjecture.

Corollary 1

For any set $X \subseteq \mathbb{R}$, the following statements are equivalent:

1. $X$ has strong measure zero;

2. every dense open set contains a translate of $X$;

3. every dense $G_4$ set contains a translate of $X$.

Fickett’s question for dense open sets

The next theorem answers the Fickett’s question for dense open sets.

Theorem 5

For every set $X \subseteq \mathbb{R}$, the following statements are equivalent:

1. for every dense open set $D \subseteq \mathbb{R}$ there exist $a, b \in \mathbb{R}$, $a > 0$, with $aX + b \subseteq D$;

2. $X$ is the union of a bounded set and a strong measure zero set.

References
