Fractional Cauchy problems with almost sectorial operators

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Introduction

The sectorial operators have been studied extensively during the last 50 years, both in the abstract setting and in its applications for partial differential equations.
Recall that a sectorial operator A, defined in a Banach space, is a closed and densely defined operator whose spectrum lies in a sector $S_\omega = \{z \in \mathbb{C}\backslash\{0\} : |\arg z| \le \omega\} \cup \{0\}$ for some $0 \le \omega < \pi/2$, and whose the resolvent operator satisfies the estimate

$$\|(z-A)^{-1}\| \le M|z|^{-1} \; \forall z \in \mathbb{C}\backslash S_\omega. \tag{1}$$

Many important elliptical differential operators belong to the class of sectorial operators, especially when they are defined in some Lebesgue space. However, if we consider these elliptic operators defined in a more regular space, like the Hölder continuous functions, then the estimative (1) does not hold; we only manage to obtain a "more weakly" inequality. This new behavior allows us to prove the existence of a semigroup that is singular at $t=0$. To clarify the ideas presented above, we intruduce the almost sectorial operators.

Definition 1. *Let $-1<\gamma<0$, $0<\omega<\pi/2$ and X is a Banach space over $\mathbb{C}$. By Θ_ω^γ we denote the family of all linear closed operators $A : D(A) \subset X \to X$ which satisfies:*

i) $\sigma(A) \subset S_\omega = \{z \in \mathbb{C} \setminus \{0\}; |\arg z| \le \omega\} \cup \{0\}$;

ii) there exists $C_\omega > 0$ such that:

$$\|(z-A)^{-1}\| \le C_\omega |z|^\gamma, \; \forall z \in \mathbb{C}\backslash S_\omega.$$

A linear operator A will be called an almost sectorial operator on X, if $A \in \Theta_\omega^\gamma(X)$.

Now consider the fractional Cauchy problem:

$$\begin{cases} {}_cD_t^\alpha u(t) + Au(t) = f(t,u(t)), \; t>0, \\ u(0) = u_0 \in X, \end{cases} \tag{2}$$

where X is a Banach space over $\mathbb{C}$, $\alpha \in (0,1)$, $A : \mathcal{D}(A) \subset X \to X$ is a almost sectorial operator, $f : [0,\infty) \times X \to X$ is a continuous function and cD_t^α is the Caputo fractional derivative where

$${}_cD_t^\alpha u(t) := \frac{d}{dt}\left\{\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}u'(s)ds\right\}$$

and u is a suitable function.

Some properties

Let $\alpha, \beta > 0$, $z \in \mathbb{C}$ and $E_{\alpha,\beta} : \mathbb{C} \to \mathbb{C}$ the generalized Mittag-Leffler special function defined by

$$E_{\alpha,\beta} := \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n + \beta)}.$$

Now consider also the Wright-type function, with $0<\alpha<1$ and $z \in \mathbb{C}$, given by

$$\psi_\alpha(z) := \sum_{n=0}^\infty \frac{(-z)^n}{n!\Gamma(-\alpha n + 1 - \alpha)}.$$

Thus we have the following properties:

i) $\displaystyle\int_0^\infty \psi_\alpha(t)e^{-zt}dt = E_\alpha(-z)$, $z \in \mathbb{C}$;

ii) $\displaystyle\int_0^\infty \alpha t\psi_\alpha(t)e^{-zt}dt = e_\alpha(-z)$, $z \in \mathbb{C}$.

For more details see [1].

Theorem 2. *Assume that $\alpha \in (0,1)$ and $A \in \Theta_\omega^\gamma(X)$. Then the operators $\{S_\alpha(t) : t>0\}$ and $\{P_\alpha(t) : t>0\}$, given by*

$$S_\alpha(t) := \frac{1}{2\pi i}\int_\Gamma E_\alpha(-zt^\alpha)(z-A)^{-1}dz,$$

$$P_\alpha(t) := \frac{1}{2\pi i}\int_\Gamma e_\alpha(-zt^\alpha)(z-A)^{-1}dz,$$

where the contour Γ starts and ends at $-\infty$, encircles the disc $|\lambda| \le |z|^{\frac{1}{\alpha}}$ and is oriented counter-clockwise, are bounded linear operators.

Existence and Uniqueness

The appropriate definition for the concept of solution to problem (2) is:

Definition 3. *Let $u \in C((0,T];X)$, with $T>0$, be a continuous function. We say that u is a mild solution of (2) if*

$$u(t) = S_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1}P_\alpha(t-s)f(s,u(s))ds, \; t \in (0,T].$$

The existence and uniqueness of solution to this problem was already studied (see [1,2,3,4]), and can be enunciated as follows:

Theorem 4. *Let $A \in \Theta_\omega^\gamma(X)$ with $-1<\gamma<-1/2$ and $0<\omega<\pi/2$. Suppose that the nonlinear function $f : (0,T] \times X \to X$ is continuous with respect to t end assume the existence of constants $M, N > 0$ satisfying*

$$\|f(t,x) - f(t,y)\| \le M(1 + \|x\|^{\nu-1} + \|y\|^{\nu-1})\|x-y\|,$$

with

$$\|f(t,x)\| \le N(1+\|x\|^\nu),$$

for all $t \in (0,T]$, for each $x,\; y \in X$ and $\nu \in \left[1, \frac{\gamma}{1+\gamma}\right)$. Then, for every $u_0 \in X$, there exists a $T_0>0$ such that the problem (2) has a unique mild solution defined on $(0,T_0]$.

Proof of Theorem 4

Proof. We will begin by defining for $r>0$ the complete metric space:

$$F_r(T,u_0) = \{u \in C((0,T];X); \; \rho_T(u, S_\alpha(t)u_0) \le r\},$$

where

$$\rho_T(u_1,u_2) = \sup_{t\in(0,T]} \|u_1(t) - u_2(t)\|.$$

Let us prove the hypotheses of Banach fixed-point theorem for function Γ^α, which is given by

$$(\Gamma^\alpha u)(t) = S_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1}P_\alpha(t-s)f(s,u(s))ds, \; u \in F_r(T_0,u_0).$$

Since f is continuous and $S_\alpha(t)$ is uniformly continuous for $t>0$, then

$$(\Gamma^\alpha u)(t) \in C((0,T];X) \quad \text{and} \quad \|(\Gamma^\alpha u)(t) - S_\alpha(t)u_0\| \le r.$$

Then, for any $u,v \in F_r(T_0,u_o)$, with $0<T_0\le T$, we have

$$\|(\Gamma^\alpha u)(t) - (\Gamma^\alpha v)(t)\| \le \frac{1}{2}\rho_{T_0}(u,v).$$

Therefore Γ^α is a contraction on $F_r(T_0,u_0)$. Then, by Banach fixed-point theorem, we have that Γ^α has a unique fixed point $u \in F_r(T_0,u_0)$, which is a mild solution to problem (2) on $(0,T_0]$. □

More Regular Solutions

Before we discuss more regular solutions, we introduce the fractional powers of an almost sectorial operator.

Definition 5. *Let $A \in \Theta_\omega^\gamma$ and $\beta > 1+\gamma$. Then we define*

$$A^{-\beta} = \frac{1}{\Gamma(\beta)}\int_0^\infty s^{\beta-1}T(s)\,ds$$

where $\{T(t) : t \ge 0\}$ is the semigroup generated by $-A$.

A fundamental result that allows us to better understand these operators is given bellow.

Proposition 6. *Let $A \in \Theta_\omega^\gamma$ and $\beta > 1+\gamma$. Then the operator $A^{-\beta} \in \mathcal{L}(X)$ and it is injective. Moreover, if $\beta,\delta > 1+\gamma$ then*

$$A^{-\beta}A^{-\delta} = A^{-(\beta+\delta)}.$$

Definition 7. *Supported by Proposition 6 we can define the fractional power of A, for any $\beta > 1+\gamma$, as*

$$A^\beta : \mathcal{D}(A^\beta) \subset X \to X,$$

where $\mathcal{D}(A^\beta) := \mathcal{R}(A^{-\beta})$ and $A^\beta := (A^{-\beta})^{-1}$.

Finally a very important theorem that improves, under certain conditions, the regularity of the mild solution obtained in Theorem 4.

Theorem 8. *Let $A \in \Theta_\omega^\gamma(X)$ with $-1<\gamma<-1/2$, $0<\omega<\pi/2$ and $u_0 \in X^1$. Assume the existence of a continuos function $M_f : \mathbb{R}^+ \to \mathbb{R}^+$ and a constant $N_f > 0$ such that $f : (0,T] \times X^1 \to X^1$ satisfies:*

$$\|f(t,x) - f(t,y)\|_{X^1} \le M_f(r)\|x-y\|_{X^1}$$

and

$$\|f(t, S_\alpha(t)u_0)\|_{X^1} \le N_f(1 + t^{-\alpha(1+\gamma)}\|u_0\|_{X^1}),$$

for all $0<t\le T$ and for each $x,\; y \in X^1$ satisfying

$$\sup_{t\in(0,T]} \|x(t) - S_\alpha(t)u_0\|_{X^1} \le r,$$

and

$$\sup_{t\in(0,T]} \|y(t) - S_\alpha(t)u_0\|_{X^1} \le r.$$

Then there is $T_0>0$ such that the problem (2) has a unique mild solution defined on $(0,T_0]$.

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