Fractional Cauchy problems with almost sectorial operators

Marduck M. Henao
Universidade federal de Santa Catarina - UFSC, Brasil
marduckmehoya@gmail.com

Introduction

The sectorial operators have been studied extensively during the last 50 years, both in the abstract setting and in its applications for partial differential equations. Recall that a sectorial operator $A$ defined in a Banach space, is a closed and densely defined operator whose spectrum lies in a sector $S_\alpha = \{ x \in \mathbb{C} \mid |\mathfrak{R}(x)| \leq \alpha |\mathfrak{I}(x)| \}$ for some $0 \leq \alpha < 1/2$ and whose resolvent operator satisfies the estimate

$$|x - A^{-1}| \leq M|x|^{-\alpha}, \quad x \in C S_\alpha.$$  

(1)

Many important elliptical differential operators belong to the class of sectorial operators, especially when they are defined in some Lebesgue space. However, if we consider these elliptical operators defined in a more regular space, like the Holder continuous functions, then the estimate (1) does not hold; we only manage to obtain a “more weakly” inequality. This new behavior allows us to prove the existence of a semigroup that is singular at $t = 0$. To clarify the ideas presented above, we introduce the almost sectoral operators.

Definition 1. Let $-1 < \gamma < 0$, $0 < \sigma < \pi/2$ and $X$ is a Banach space over $\mathbb{C}$. By $B_{\sigma}^\gamma$ we denote the family of all linear closed operators $A : D(A) \subset X \to X$ which satisfies:

- $i(0) A \subset S_\sigma \setminus \{0\}$, $\sigma \in (0,1)$; $\sigma \leq \alpha \leq \sigma (2)$
- $i(0) A \subset S_\sigma \setminus \{0\}$, $\alpha \leq \sigma \leq \alpha (2)$

Definition 2. A sectorial operator $A$ will be called an almost sectorial operator on $X$, if $A \in B_{\sigma}^\gamma(X)$.

Proof of Theorem 4

Proof We will begin by defining for $r > 0$ the complete metric space: $\mathcal{F}_r(T, u_0) = \{ u \in C(0, T), \mathbb{S}_r(u) : \mathcal{S}_r(u)(t) \to U \}$, where

$$\mathcal{S}_r(u)(t) = \sup_{\nu \neq 0} \left| \langle u(t), \nu \rangle - t \langle \nu, \nu \rangle \right|.$$  

Let us prove the hypotheses of Banach fixed-point theorem for function $F$, which is given by $F(t)(u_0) = \mathcal{S}_r(u_0) + \int_0^t (t - s)^{-\gamma}P(t - s)\langle u(s), u(s) \rangle ds, \quad w \in F(T, u_0).$  

Since $f$ is continuous and $S_\gamma$ is uniformly continuous for $t > 0$, then $\left| F(t)(u_0) - S_r(u_0) \right| \to 0$.  

Then, for any $u_0 \in F(T, u_0)$, we have

$$\left| F(t)(u_0) - S_r(u_0) \right| \to 0$$  

Therefore $F$ is a contraction on $F(T, u_0)$.  

Then, by Banach fixed-point theorem, we have that $F$ has a unique fixed point $u \in F(T, u_0)$, which is a mild solution to problem (2) on $[0, T]_0$.

More Regular Solutions

Before we discuss more regular solutions, we introduce the fractional powers of an almost sectorial operator.

Definition 5. Let $A \in B_{\sigma}^\gamma$ and $\beta > 1 + \gamma$. Then we define $\frac{\partial}{\partial t} \alpha A = \frac{1}{(1 - \alpha)\beta} \int_0^t \frac{1}{\beta T} R(t)(\alpha \mathbb{S}(U)) dt$  

where $R(t) \in \mathcal{S}(U)$ is the semigroup generated by $A$.  

A fundamental result that allows us to better understand these operators is given below:  

Proposition 6. Let $A \in B_{\sigma}^\gamma$ and $\beta > 1 + \gamma$. Then the operator $\frac{\partial}{\partial t} \alpha A = \mathcal{S}(U)$ and it is injective. Moreover, if $\beta > 1 + \gamma$ then $\frac{\partial}{\partial t} \alpha A^{\beta - 1} = A^{\beta - 1}$.  

Finally, a very important theorem that improves, under certain conditions, the regularity of the mild solution obtained in Theorem 4.

Theorem 7. Let $A \in B_{\sigma}^\gamma$ with $-1 < \gamma < -1/2$, $0 < \sigma < \pi/2$ and $A \in C_0^0$.  

Assume the existence of a continuous function $M_1 : \mathbb{R}^+ \to \mathbb{R}^+$ and a constant $N_3 > 0$ such that $f \in L^1(0, T) \times X \to X$ satisfies

$$\left| \langle f(t), \nu \rangle - f(t) \langle \nu, \nu \rangle \right| \leq M_1(t) \left| f(t) - f(t^-) \right| \gamma.$$  

and

$$\left| f(t), \mathcal{S}_r(u_0)(t) \right| \leq N_3 \left| f(t) - f(t^-) \right| \gamma.$$  

for all $0 \leq t < T$ and for each $y \in X$. Then for $\nu \in X$ satisfies

$$\sup_{t \in [0, T]} \left| \langle f(t), \mathcal{S}_r(u_0)(t) \rangle \right| \leq r,$$

and

$$\sup_{t \in [0, T]} \left| \langle f(t), \nu \rangle \right| \leq r.$$  

Then there is $T_0 > 0$ such that the problem (2) has a unique mild solution defined on $[0, T_0]_0$.

Existence and Uniqueness

The appropriate definition for the concept of solution to problem (2) is

Definition 3. Let $u \in C(0, T), X$, with $T > 0$, be a continuous function. We say that $u$ is a mild solution of (2) if

$$u(t) = S_{\gamma}(u_0) + \int_0^t \mathcal{S}_r(u)(t) dt, \quad t \in [0, T].$$

The existence and uniqueness of solutions to this problem was already studied (see [1,2,4,5]), and can be summarized as follows:  

Theorem 4. Let $A \in B_{\sigma}^\gamma(X)$ with $-1 < \gamma < -1/2$ and $\sigma < \pi/2$, $\alpha > 0$, and $\gamma > 0$. Suppose that the continuous function $F : \mathbb{R}_+ \times X \times X$ satisfies with respect to $x$ and assume the existence of constants $M_1, M_2 > 0$ satisfying

$$\left| F(t, x), x \right| \leq M_1 \left| x \right|^{\gamma-1} \left| x \right|^{\gamma-1} \left| x \right| + q.$$  

with

$$\left| F(t, x), x \right| \leq M_1 \left| x \right|^{\gamma-1} \left| x \right|^{\gamma-1} \left| x \right| + q.$$  

for all $t \in [0, T]$ for each $y \in X$ and $x \in \left[ \frac{1}{1-x} \right]$. Then for every $u_0 \in X$, there exists a $T_0 > 0$ such that the problem (2) has a unique mild solution defined on $[0, T_0]_0$.

Acknowledgement

I would like to thanks Prof. Dr. Paulo M. Carvalho Neto. Also, I want to thanks the department of Pure and Applied Mathematics of the Federal University of Santa Catarina (UFSC) and CNPq

Bibliography