Manifolds of **constant sectional curvature** can be **characterized** by certain properties of their **spherical curves**.

Characterization of manifolds of constant curvature by spherical curves

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Introduction

- *Space forms* are characterized in terms of geodesic spheres: e.g., all their spheres are umbilical.
- In $\mathbb{E}^n$, $\mathbb{H}^n(r)$, and $\mathbb{S}^n(r)$ the behavior of spherical curves is well understood.
- We show how to characterize space forms by studying the behavior of spherical curves.

We provide a *new proof* for the link "const. curvature $\Leftrightarrow$ umbilicity of spheres" [Kulkarni 1975], but using rotation minimizing (RM) frames:
Definition. A normal vector field V is RM along α if and only if $\exists\,\lambda$: $\nabla_{\alpha'} V = \lambda\alpha'$.

Space forms and umbilical spheres

Theorem 1. M^{m+1} is a space form $\Leftrightarrow$ every sphere $G(p,R)$ in M^{m+1}, R small enough, is totally umbilical.

- Comes from the fact that the normal ξ is RM along *any* α in $G(p,R)$ when the shape operator $S(X) \equiv -\nabla_X\xi$ satisfies for all $X \in \mathfrak{X}(G(p,R))$, $S(X) = \lambda X$, i.e., when $S(x) = H\,\mathrm{Id}$.
- Given X_q, Y_q, we then exploit the 2-surface defined by $f(u,s) = \exp_p(uV(s))$ s.t. $q = f|_{(R,0)}$ and $\frac{\partial f}{\partial s}|_{(R,0)} = X_q$, $\frac{\partial f}{\partial u}|_{(R,0)} = Y_q$. Umbilicity means $\nabla_X Y = -\lambda X$.

Space forms and spherical curves

From the umbilicity of spheres, we have the following characterizations:

Theorem 2. M^{m+1} is a space form $\Leftrightarrow$ for all α in $G(p,R)$, R sufficiently small, $\sum_{i=1}^{m} a_i\kappa_i(s) = H(s)$, $\xi(s) = \sum_{i=1}^{m} a_i\,\mathbf{n}_i(s)$, *where H is the mean curvature, a_i are constants, and $\{\alpha', \mathbf{n}_1, \ldots, \mathbf{n}_m\}$ is an RM frame along α.*
Theorem 3. M^{m+1} is a space form if and only if every curve $\alpha(s)$ in $G(p,R)$, R small, satisfies
$$\kappa^2(s) \geq H^2(\alpha(s)),$$
where H is the mean curvature and κ is the curvature function of $\alpha(s)$ in M^{m+1}. Also, equality holds for geodesics only.
Theorem 4. M^3 is a space form if and only if every closed curve α in $G(p,R)$, R small, has a vanishing total torsion $\oint_\alpha \tau$.

Remarks. (A) In a generic manifold, Thrs. 2, 3, and 4 may be adapted to characterize totally umbilical hypersurfaces.
(B) Theorems 1, 2, and 3 are also valid for semi-Riemannian manifolds.

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Reference

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