Limit theorems for dependent Bernoulli processes and related random walks

Lucas Roberto de Lima & Cristian Favio Coletti

Universidade Federal do ABC

lucas.roberto@ufabc.edu.br, cristian.coletti@ufabc.edu.br

impa Instituto de Matemática Pura e Aplicada

Abstract

We study limit theorems for a class of Bernoulli random processes where the success probability of a trial, conditional on the previous trials, depends on the total number of successes already achieved. We show sufficient conditions under which it is possible to prove the strong law of large numbers, the central limit theorem and the law of iterated logarithm for partial sums of the Bernoulli random variables via martingale techniques. Moreover, we apply the results and techniques in some random walks with unbounded memory.

Introduction

We present a similar model from the ones studied in [3] and [4], where we extend the hypothesis obtaining the same results. This modification allows new applications, such as in the elephant random walk.

Let $\{X_n\}_{n\geq 1}$ be a sequence of random variables with $X_k \sim \mathrm{Ber}(p_k)$ and $p_1 \in \,]0,1[$ satisfying

$$\mathbb{P}(X_{n+1}=1|\mathcal{F}_n) = \theta_n + d_n\frac{S_n}{n} \tag{1}$$

where $\{\theta_n\}_{n\geq 1}$ and $\{d_n\}_{n\geq 1}$ are sequences of real numbers such that $d_n \in \,]-1,1[$, $(\theta_n + d_n x) \in [0,1]$ for all $x \in [0,1]$ and all $n \in \mathbb{N}^*$; $\mathcal{F}_n = \sigma(X_1,\dots,X_n)$ the σ-algebra generated by $X_1,\dots,X_n$; S_n is the partial sum $\sum_{i=1}^n X_i$. Define the sequence $\{a_n\}_{n\geq 1}$ by

$$a_n = \prod_{k=1}^{n-1}\left(1+\frac{d_k}{k}\right).$$

Let $\{A_n\}_{n\geq 1}$ and $\{s_n\}_{n\geq 1}$ be sequences of positive real numbers such that

$$A_n^2 = \sum_{k=1}^{n}\frac{1}{a_k^2} \quad \text{and} \quad s_n^2 = \sum_{k=1}^{n}\frac{p_k(1-p_k)}{a_k^2}.$$

Define $T_n = \frac{S_n - \mathbb{E}[S_n]}{a_n}$, then $\{T_n, \mathcal{F}_n, n \geq 1\}$ is a martingale and we can use martingale limit theory for the associated martingale difference in order to prove the limit theorems.

1 Limit Theorems

Theorem (Strong Law of Large Numbers). *Consider the model (1), then*

$$\lim_{n\to\infty}\frac{S_n - \mathbb{E}[S_n]}{n} = 0 \quad a.s.$$

if, and only if,

$$\sum_{k=1}^{\infty}\frac{1-d_k}{k+1} = \infty.$$

Theorem (Central Limit Theorem). *Let $\{X_n\}_{n\geq 1}$ be a sequence of random variables satisfying the model (1). If $\lim_{n\to\infty} s_n = \infty$, $\lim_{n\to\infty} a_n s_n = \infty$ and $\{A_n/s_n\}_{n\geq 1}$ is bounded, then*

$$\frac{S_n - \mathbb{E}[S_n]}{a_n s_n} \xrightarrow{D} N(0,1).$$

Theorem (Law of Iterated Logarithm). *Let $\{X_n\}_{n\geq 1}$ be a sequence of random variables satisfying the model (1). If $\lim_{n\to\infty} s_n = \infty$, $\lim_{n\to\infty} a_n s_n = \infty$ and $\{A_n/s_n\}_{n\geq 1}$ is bounded, then*

$$\limsup_{n\to\infty} \pm\frac{S_n - \mathbb{E}[S_n]}{a_n s_n\sqrt{\ln(\ln(s_n))}} = \sqrt{2} \quad a.s.$$

2 The elephant random walk

The elephant random walk is a discrete-time random walk on $\mathbb{Z}$, where, at each discrete time step, the elephant can remember all previous steps and decides to move one step to the right or to the left depending on the history of the process. We represent its position at time $n \in \mathbb{N}$ by the random variable χ_n, defined by

$$\chi_{n+1} = \chi_n + \eta_{n+1}$$

where $\eta_{n+1} = \pm 1$, we define the model as follows:

(i) starting at $\chi_0 \in \mathbb{Z}$, the elephant moves to the right with probability $q \in [0,1]$ and moves to the left with probability $1-q$, *i.e.*,

$$\mathbb{P}(\eta_1 = 1) = q \ \text{ and } \ \mathbb{P}(\eta_1 = -1) = 1-q.$$

(ii) at time step $n+1$, a number $n' \in \{1,2,\dots,n\}$ is randomly chosen with probability $1/n$.

(iii) for a given $p \in [0,1]$, we have $\mathbb{P}(\eta_{n+1} = \eta_{n'}) = p$ and $\mathbb{P}(\eta_{n+1} = -\eta_{n'}) = 1-p$.

We can set $\chi_0 = 0$ w.l.o.g., then $\chi_{n+1} = \sum_{k=1}^n \eta_k$. Since $\eta_n = 2X_n - 1$, where $\{X_1\}_{n\geq 1}$ is a Bernoulli process and we can check that

$$\mathbb{P}(X_{n+1}=1|\mathcal{F}_n) = 1-p+(2p-1)\frac{S_n}{n},$$

so the results from the model (1) can be applied in the elephant random walk. Consider a_n and s_n as defined in (1), then we state the following theorems.

Theorem (Strong Law of Large Numbers). *Consider the elephant random walk. If $p \in \,]0,1[$, then*

$$\lim_{n\to\infty}\frac{\chi_n - \mathbb{E}[\chi_n]}{a_n} = 0 \quad a.s.$$

Theorem (Central Limit Theorem). *Let $\{\chi_n\}_{n\geq 1}$ be the elephant random walk with $p \in \,]0,\frac{3}{4}]$, then*

$$\frac{\chi_n - \mathbb{E}[\chi_n]}{2a_n s_n} \xrightarrow{D} N(0,1).$$

Theorem (Law of Iterated Logarithm). *Let $\{\chi_n\}_{n\geq 1}$ be the elephant random walk with $p \in \,]0,\frac{3}{4}]$, then*

$$\limsup_{n\to\infty} \pm\frac{\chi_n - \mathbb{E}[\chi_n]}{2a_n s_n\sqrt{\ln(\ln(s_n))}} = \sqrt{2} \quad a.s.$$

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