Structure and classification of Lie algebras constructed from $\mathfrak{gl}_{m|n}$ using the derived bracket

Luan F. Oliveira[1]

Supervisor : **Elizaveta Vishnyakova**[2]

Federal University of Minas Gerais - UFMG

Mathematics Undergraduate Program[1], Institute of Exact Sciences[2]

luanmat@ufmg.br[1], vishnyakovae@googlemail.com[2]

Abstract

Our poster is devoted to a classification of Lie algebras obtained from the Lie superalgebra $\mathfrak{g} = \mathfrak{gl}_{m|n}$ over $(\mathbb{C})$ using the derived bracket construction:

$$\llbracket X, Y\rrbracket_B = [X,[B,Y]],$$

where $X,\ Y \in \mathfrak{g}_{-1}$ and $B \in \mathfrak{g}_1$. Moreover, we obtained Levi-Malcev decomposition of these Lie algebras.

Introduction

The idea of "derived bracket" came from a graded Lie algebra (or superalgebra) structure together with an odd derivation of square zero, producing a graded bracket of opposite parity. While they are not Lie brackets on the algebra itself, they give rise to a Lie algebra structure on some of its subspaces and quotients. Kosmann-Schwarzbach, in [3], created a definition attempting to generalize this notion. As an example of its importance, the derived bracket allowed Roytenberg in [4] to discover an alternative description for Courant algebroid and Courant bracket of Liu, Weinstein and Xu.

A Lie superalgebra is a generalization of a Lie algebra to the $\mathbb{Z}_2$-graded case. We use a Lie superalgebra $\mathfrak{gl}_{m|n}$ over $\mathbb{C}$ from Kac's classification [1] to construct and classify a family of Lie algebras.

Basic tools

Definition 1. *A superspace V is a $\mathbb{Z}_2$-graded vector space ($V = V_{\bar 0} \oplus V_{\bar 1}$). A superalgebra $\mathcal{A} = \mathcal{A}_{\bar 0} \oplus \mathcal{A}_{\bar 1}$ is a superspace over $\mathbb{C}$ such that for every $a, b \in \mathbb{Z}_2$:*

$$\mathcal{A}_a \mathcal{A}_b \subset \mathcal{A}_{a+b}.$$

Definition 2. *A Lie superalgebra $\mathfrak{L}$ is a superalgebra with a Lie superbracket operation $[\cdot\, ,\cdot\,]$, satisfying:*
i) $[a,b] = -(-1)^{(deg\ a)(deg\ b)}[b,a]$;
ii) $[a,[b,c]] = [[a,b],c] + (-1)^{(deg\ a)(deg\ b)}[b,[a,c]]$.

Definition 3. *Let G be an abelian group and $V = \bigoplus_{\alpha \in G} (V_\alpha)$ be a G-graded space. Then the associative algebra (End V, $\circ$), where "$\circ$ "is the usual composition, possesses a G-grading defined by:*

$$\text{End } V = \bigoplus_{\alpha \in G} (\text{End } V_\alpha),$$

$$\text{End } V_\alpha = \{f \in \text{End } V_\alpha \mid f(V_s) \subseteq V_{s+\alpha}\}.$$

In particular, for $G = \mathbb{Z}_2$ we obtain the associative superalgebra:

$$\text{End } V = \text{End } V_{\bar 0} \oplus \text{End } V_{\bar 1}.$$

Definition 4. *Let V be a superspace such that $dim\ V_{\bar 0} = m$ and $dim\ V_{\bar 1} = n$. The Lie superalgebra $\mathfrak{gl}_{m|n}$ is defined as the superspace End $V = \mathfrak{gl}(V)$ endowed with the following Lie superbracket $[X,Y] = X \circ Y - Y \circ X$. If bases of $V_{\bar 0}$ and $V_{\bar 1}$ are fixed, we can identify $\mathfrak{gl}(V)$ with Lie superalgebras of matrices of the form:*

$$\begin{array}{cc} & \begin{array}{cc} m & n \end{array} \\ \begin{array}{c} m \\ n \end{array} & \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{array}.$$

Remark 1. *The Lie superalgebra $\mathfrak{gl}_{m|n}$ admits the following $\mathbb{Z}$-grading:*

$$\mathfrak{gl}_{m|n} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1,$$

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in \mathfrak{g}_0,\ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \in \mathfrak{g}_{-1},\ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}_1.$$

Definition 5. *A radical $\mathfrak{r}$ of a finite-dimensional Lie algebra L is the maximal solvable ideal $\mathfrak{r}$ of L.*
A Lie algebra L is called semisimple if its radical $\mathfrak{r}$ is equal to zero.

Definition 6. *If $L = L_{-1} \oplus L_0 \oplus L_1$ is a graded Lie superalgebra over $\mathbb{C}$ with bracket $[\cdot,\cdot]$ and $B \in L_1$ such that $[B,B] = 0$, we can define the following bilinear map $\llbracket \cdot,\cdot \rrbracket_{(B)} : V \otimes V \to V$:*

$$\llbracket x, y\rrbracket_{(B)} = [x,[B,y]],$$

for x and $y \in L_{-1}$. We call these new bracket the derived bracket of $[\cdot,\cdot]$ by B.

Lemma 1. *Let $B \in Mat_{m\times n}(\mathbb{C})$, rank B=r. Then there exist two invertible matrices P_m and P_n such that $P_m B P_n = B'$, where $B' = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$. Here I_r is the identity matrix of size r.*

Proof: This is a standard result from linear algebra.

Lemma 2. *Let L be a finite dimensional Lie algebra over $\mathbb{C}$. Let $\mathfrak{r}$ be its radical and consider a solvable ideal $\mathfrak{a}$. Then:*
a) $L/\mathfrak{r}$ is semisimple.
b) A quotient $L/\mathfrak{a}$ is semisimple if and only if $\mathfrak{r} = \mathfrak{a}$.

Proof:
a) Let $\pi : L \to L/\mathfrak{r}$ be the canonical map and J be a solvable ideal of $L/\mathfrak{r}$.
Then $U = \pi^{-1} J$, $\mathfrak{r} \subseteq U$ and $U/\mathfrak{r} \cong J$. As U is solvable and $\mathfrak{r}$ is maximal, $U/\mathfrak{r} = 0$, therefore, $J = 0$.
b) As $L/\mathfrak{a}$ is semisimple, $\mathfrak{r}/\mathfrak{a} = 0$ then $\mathfrak{r} = \mathfrak{a}$.
Conversely, if $\mathfrak{r} = \mathfrak{a}$ then, by the item "a", $L/\mathfrak{a}$ is semisimple.

Theorem 1. *(Levi) Let L be a finite dimensional Lie algebra. If L is not solvable, then there exists a semisimple subalgebra $\mathfrak{s}$ of L such that:*

$$L = \mathfrak{s} \ltimes \mathfrak{r},\ \mathfrak{s} \cong L/\mathfrak{r}.$$

Proof: See [2] for more details.

Theorem 2. *(Malcev) Let $\mathfrak{s}$ and $\mathfrak{s}'$ be semisimple subalgebras of L with $L = \mathfrak{s} \ltimes \mathfrak{r} = \mathfrak{s}' \ltimes \mathfrak{r}$. Then there exists an automorphism σ of L such that $\sigma(\mathfrak{s}) = \mathfrak{s}'$.*

Proof: See [2] for more details.

Results

Lemma 3. *Let $B \in \mathfrak{g}_1$ and $X, Y \in \mathfrak{g}_{-1}$ in the form of block matrices x_{ij} and y_{ij}, $i,j = \{1,2\}$. Then:*

$$[X[B,Y]] = \begin{pmatrix} 0 & 0 \\ XBY - YBX & 0 \end{pmatrix},$$

$$XBY - YBX = \begin{pmatrix} x_{11}{**}y_{11} - y_{11}{**}x_{11} & x_{11}{**}y_{12} - y_{11}{**}x_{12} \\ x_{21}{**}y_{11} - y_{21}{**}x_{11} & x_{21}{**}y_{12} - y_{21}{**}x_{12} \end{pmatrix}.$$

Proof: A direct calculation and the result from Lemma 1 shows the proof.

Theorem 3. *Let V be a superspace, $\mathfrak{gl}(V) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_1$ and $\llbracket \cdot,\cdot \rrbracket_B$ be the derived bracket by B.*
Then: $(\mathfrak{g}_{-1}, \llbracket \cdot,\cdot \rrbracket_B)$ is a Lie algebra and its center is:

$$Z = \begin{pmatrix} 0 & 0 \\ 0 & y_{22} \end{pmatrix}.$$

Proof: A direct calculation shows that $(\mathfrak{g}_{-1}, \llbracket\ ,\ \rrbracket_B)$ respects anticommutativity and the Jacobi identity.
For the center, let $y \in Z$ and $x \in \mathfrak{gl}_{m|n}$. Using the formula of Lemma 3, we see that $[x[B,y]] = 0$. Now consider $w \in \mathfrak{gl}_{m|n}$, such that $w \notin Z$. It is easy to see that $[x[B,w]] \neq 0$ for all x. Therefore, Z is the center of $(\mathfrak{g}_{-1}, \llbracket\ ,\ \rrbracket_B)$.

Theorem 4. *Let $(\mathfrak{g}_{-1}, \llbracket \cdot,\cdot \rrbracket_B)$ be a Lie algebra, $B \in \mathfrak{g}_1$, rank $B = r$ and B' as in Lemma 1. Then there is an isomorphism of Lie algebras:*

$$(\mathfrak{g}_{-1}, \llbracket \cdot,\cdot \rrbracket_B) \xrightarrow{\sim} (\mathfrak{g}_{-1}, \llbracket \cdot,\cdot \rrbracket_{B'}).$$

Proof: Consider the matrix $P = \begin{pmatrix} P_m & 0 \\ 0 & P_n^{-1} \end{pmatrix}$, where P_m and P_n are from Lemma 1.
Let $\phi_P : \mathfrak{gl}_{m|n} \to \mathfrak{gl}_{m|n}, X \mapsto PXP^{-1}$;
$\pi : \mathfrak{gl}_{m|n} \to \mathfrak{g}_{-1}$ is the natural projection .
We choose to demonstrate the theorem by constructing a diagram of $\mathfrak{gl}_{m|n}$ and $\mathfrak{g}_{-1}$, using the mappings defined above. To achieve that we need to prove that each $\mathfrak{g}_{-1}$, $\mathfrak{g}_0$ and $\mathfrak{g}_1$ are invariant by ϕ_P.

Let $T = \begin{array}{cc} & \begin{array}{cc} m & n \end{array} \\ \begin{array}{c} m \\ n \end{array} & \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \end{array}$, $T \in \mathfrak{gl}_{m|n}$.

$\phi_P(T) = \begin{pmatrix} P_m & 0 \\ 0 & P_n^{-1} \end{pmatrix} \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \begin{pmatrix} P_m^{-1} & 0 \\ 0 & P_n \end{pmatrix} = \begin{pmatrix} P_m X P_m^{-1} & P_m Y P_n \\ P_n^{-1} Z P_m^{-1} & P_n^{-1} W P_n \end{pmatrix}$.

Observe that each subspace have a specific size, then $(\phi_P(X),\ \phi_P(W)) \subseteq \mathfrak{g}_0$, $\phi_P(Y) \subseteq \mathfrak{g}_1$ and $\phi_P(Z) \subseteq \mathfrak{g}_{-1}$.
Now we can construct the following diagram:

$$\begin{array}{ccc} \mathfrak{gl}_{m|n} & \xrightarrow{\phi_P} & \mathfrak{gl}_{m|n} \\ \pi \downarrow & & \downarrow \pi \\ \mathfrak{g}_{-1} & \xrightarrow{\phi_P} & \mathfrak{g}_{-1} \end{array}.$$

We just need to prove that ϕ_P is a homomorphism of Lie algebras, i. e.,
$\phi_P \llbracket X, Y\rrbracket = \llbracket \phi_P(X), \phi_P(Y)\rrbracket$.
Let $X, Y \in \mathfrak{g}_{-1}$.
By Lemma 3, we have:

$$\phi_p \llbracket X, Y\rrbracket_B = \begin{pmatrix} P_m & 0 \\ 0 & P_n^{-1} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ XBY - YBX & 0 \end{pmatrix} \begin{pmatrix} P_m^{-1} & 0 \\ 0 & P_n \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ P_n^{-1} XBY P_m^{-1} - P_n^{-1} YBX P_m^{-1} & 0 \end{pmatrix}.$$

Writing B as $P_m^{-1} B' P_n^{-1}$ we get:

$$\begin{pmatrix} 0 & 0 \\ (P_n^{-1} X P_m^{-1}) B' (P_n^{-1} Y P_m^{-1}) - (P_n^{-1} Y P_m^{-1}) B' (P_n^{-1} X P_m^{-1}) & 0 \end{pmatrix}$$

$= \llbracket P_n^{-1} X P_m^{-1}, P_n^{-1} Y P_m^{-1} \rrbracket_{B'} = \llbracket \phi_P(X), \phi_P(Y)\rrbracket_{B'}$.
Therefore ϕ_P is a homomorphism of Lie algebras.

Theorem 5. *Consider the Lie algebras $(\mathfrak{g}_{-1}, \llbracket \cdot,\cdot \rrbracket_{B'})$ and $r = \{0, ..., l\}$, $l = min\{m,n\}$, be the possible integer values for rank B'. Let $R = \{\begin{pmatrix} \alpha I_r & X_{12} \\ X_{21} & X_{22}. \end{pmatrix}\}$, where αI_r is a scalar matrix of size r and $S = \{\begin{pmatrix} X_{11} & 0 \\ 0 & 0. \end{pmatrix}\}$, where trace S = 0 and X_{11} is a matrix of size r .*

Then:
a) R is solvable;
b) S is semisimple;
c) $R \cong \mathfrak{r}(\mathfrak{g}_{-1})$.
d) The family of Lie algebras $(\mathfrak{g}_{-1}, \llbracket \cdot,\cdot \rrbracket_{B'})$ admits a parametrization by r.

Proof:
a) A direct calculation shows that $R^{(2)} \subseteq Z$, where Z is the center of $(\mathfrak{g}_{-1}, \llbracket \cdot,\cdot \rrbracket_{B'})$, therefore R is solvable.
b) Observe that $S \simeq \mathfrak{sl}_n$, then S is simple and only admits 0 as solvable ideal. Therefore, S is semisimple.
c) Follows directly from Lemma 2.
d) Consider $X \in (\mathfrak{g}_{-1}, \llbracket \cdot,\cdot \rrbracket_{B_1'})$ and $Y \in (\mathfrak{g}_{-1}, \llbracket \cdot,\cdot \rrbracket_{B_2'})$, where rank $B_1' = r_1$, rank $B_2' = r_2$ and $r_1 \neq r_2$. From Theorem 1, there is a decomposition such that $X = S_1 \ltimes R_1$ and $Y = S_2 \ltimes R_2$. From the fact that $S \simeq \mathfrak{sl}_n$, $dim\ \mathfrak{sl}_n = n^2 - 1$ and $r_1 \neq r_2$, we conclude that $dim\ S_1 \neq dim\ S_2$ therefore $S_1 \neq S_2$. From Theorem 2, we know that the subalgebra of this decomposition is unique up to an automorphism. In another words, the algebras associated with a specific r_i cannot be isomorphic to another algebra associated with $r \neq r_i$. Therefore, each Lie algebra $(\mathfrak{g}_{-1}, \llbracket \cdot,\cdot \rrbracket_{B_i'})$ is uniquely associated with each parameter r.

References

[1] Kac, V. G.. Lie Superalgebras, Massachusetts Institute of Technology, 1973.
[2] Goto, M., Grosshans, F. , Semisimple Lie algebras, Marcel Dekker, 1978.
[3] Kosmann-Schwarzbach, Y., Derived brackets, https://arxiv.org/abs/math/0312524, 2003.
[4] Roytenberg, D., Courant algebroids, derived brackets and even symplectic supermanifolds, https://arxiv.org/abs/math/9910078, 1999.