A problem with the Biharmonic Operator

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Abstract

This work tries an eigenvalue problem that incorporates the biharmonic operator. This one concerns to find the existence of real numbers ω and real functions u, ϕ satisfying the system

$$\begin{aligned} -\Delta u + \phi u &= \omega u \ \text{ in } \Omega \\ \Delta^2 \phi - \Delta \phi &= u^2 \ \text{ in } \Omega \end{aligned} \tag{1}$$

with the boundary and normalizing conditions

$$u = \Delta \phi = \phi = 0 \ \text{ on } \ \partial\Omega \ \text{ and } \ \int_\Omega u^2 = 1. \tag{2}$$

This problem is associated with a single particle of mass $m = \frac{2}{\hbar^2}$ moving under the influence of an arbitrary force field described by the potential energy ϕ.

Introduction

By the classic inspection the function ϕ requires necessarily belong to $H := H^2(\Omega) \cap H_0^1(\Omega)$. H is a Hilbert space with the equivalent norm induced by the inner product

$$(u, v)_H = \int_\Omega (\Delta u \Delta v + \nabla u \nabla v)\, dx.$$

Also, it is not difficult to see the functional:

$$F(u,\phi) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx + \frac{1}{2}\int_\Omega \phi u^2 dx - \frac{1}{4}\int_\Omega |\Delta \phi|^2\, dx - \frac{1}{4}\int_\Omega |\nabla \phi|^2\, dx, \tag{3}$$

on the manifold

$$M = \left\{ (u,\phi) \in H_0^1(\Omega) \times H; \|u\|_{L^2(\Omega)} = 1 \right\},$$

satisfies the Euler-Lagrange equations of the system, also F is a strongly indefinite functional, this means F is neither bounded from above nor from below. Then the usual methods of the critical points can not be directly used. To deal with this difficulty we shall reduce the study of (3) to the study of a functional of the only variable u, likewise that was done by Benci and Fortunato in [1]. Hence applying the Riesz isomorphism to the second equation in (1), when $u \in H_0^1(\Omega)$ is fixed, it has a unique solution $\phi = \phi(u) \in H$.

Aim

Theorem 0.1. *Let Ω be a bounded set in $\mathbb{R}^3$. Then there is a sequence (ω_n, u_n, ϕ_n), with $\{\omega_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$, $\omega_n \to \infty$ and u_n, ω_n are real functions, solving from* (1) *to* (3).

1 Results

Proposition 1.1. *$\omega \in \mathbb{R}$ and $(u,\phi) \in M$ is a weak solution of the system if and only if (u,ϕ) is a critical point of the functional $F|_M$ and ω as lagrangian multiplier.*

Set $\Gamma = \left\{ (u,\phi) \in H_0^1(\Omega) \times H : F'_\phi(u,\phi) = 0 \right\}$. Consider the map

$$\begin{aligned} L : B &\longrightarrow H \\ u &\longmapsto L(u) = \phi \text{ is solution of 2nd eq. in (1)} \end{aligned} \tag{4}$$

where $B = \left\{ u \in H_0^1 \ \|u\|_{L^2(\Omega)} = 1 \right\}$.

Proposition 1.2. *The map L is C^1 and Γ is the graph of L.*

Replace $L(u)$ in (3) we define

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx + \frac{1}{4}\int_\Omega |\Delta L(u)|^2\, dx + \frac{1}{4}\int_\Omega |\nabla L(u)|^2\, dx. \tag{5}$$

By proposition 1.2 $J|_B$ is C^1 and it is bounded from below.

Proposition 1.3. *Let $(u,\phi) \in M$ and $\omega \in \mathbb{R}$. The following statements are equivalent:*

1. *(u,ϕ) is a critical point of $F|_M$, having ω as lagrangian multiplier.*
2. *u is a critical point of $J|_B$ having ω as a langrangian multiplier and $\phi = L(u)$.*

Theorem 1.1. *There is a sequence $\{u_n\}$ of critical points of $J|_B$ having lagrangian multipliers $\omega_n \to \infty$.*

To prove this theorem we need some technical lemmas (general facts of critical points theory) see [2].

Lemma 1.1. *The functional $J|_B$ satisfies the Palais-Smale condition, i.e. any sequence $\{u_n\} \subset B$, such that $\{J(u_n)\}$ is bounded and $J|'_B(u_n) \longrightarrow 0$ contains a convergent subsequence in $H_0^1(\Omega)$.*

Lemma 1.2. *For any integer m there is a compact symmetric subset $K \subset B$ such that $\gamma(K) = m$.*

Lemma 1.3. *For any $b \in \mathbb{R}$ the sublevel*

$$J^b = \{u \in B; J(u) \le b\}$$

has finite genus.

Proof. [Theorem 1.1] Let k be a positive integer. By Lemma 1.3, there exist an integer $m = m(k)$ such that $\gamma(J^k) = m$; furthermore, consider the collection

$$\mathcal{A}_{m+1} = \{A \subset B; \text{ symmetric, closed with } \gamma(A) = m+1\}.$$

By Lemma 1.2 $\mathcal{A}_{m+1} \neq \emptyset$. Observe that $\gamma(A) > \gamma(J^k)$, for all $A \in \mathcal{A}_{m+1}$, then by monotonicity property of genus A is not a subset of J^k, hence $\sup A > k$, for all $A \in \mathcal{A}_{m+1}$, thereby

$$b_k = \inf\{\sup J(A); A \in \mathcal{A}_{m+1}\} \ge k.$$

By Lemma 1.1 $J|_B$ satisfies the (PS), the well known results in critical point theory guarantee that b_k is a critical value of $J|_B$. By the Lagrange Multiplier Theorem, for any n there is $\omega_n \in \mathbb{R}$ and $u_n \in B$ such that

$$J'(u_n) = \omega_n u_n \ \text{ and } \ J(u_n) = b_n > n. \tag{6}$$

Claim: $\omega_n \to \infty$ as $n \to \infty$. By the first equality in (6) and Proposition 1.3 we have $J'(u_n) = \omega_n u_n$. This can be written as follows

$$-\Delta u_n + \phi_n u_n = \omega_n u_n, \text{ where } \phi_n = L(u_n).$$

Hence we deduce

$$\frac{1}{2}\int_\Omega |\nabla u_n|^2\, dx + \frac{1}{2}\int_\Omega \phi_n u_n^2 dx = \omega_n \int_\Omega u_n^2 dx = \omega_n \tag{7}$$

From the functional (3) and the equation (7) we have

$$J(u_n) = \omega_n - \frac{1}{4}\int_\Omega |\Delta \phi_n|^2\, dx - \frac{1}{4}\int_\Omega |\nabla \phi_n|^2\, dx. \tag{8}$$

Using the second equality of (6) we get

$$\omega_n = b_n + \frac{1}{4}\int_\Omega |\Delta \phi_n|^2\, dx + \frac{1}{4}\int_\Omega |\nabla \phi_n|^2\, dx, \tag{9}$$

for $b_n > n$ in (6) thereby $\omega_n \to \infty$ as $n \to \infty$. □

References

[1] V. Benci and D. Fortunato. An eigenvalue problem for the schrödinger-maxwell equations. *Topological Methods in Nonlinear Analysis*, 11:283–293, 1998.

[2] P. H. Rabinowitz. Variational methods for nonlinear eigenvalue problems. In *Proc. CIME*, 1974.

Gratefulness

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