Sobolev Type Inequality for Intrinsic Varifolds

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Abstract

We present a Sobolev type inequality for varifolds intrinsically defined in a natural way, as a consequence of a generalized monotonicity formula for this kind of varifolds and avoiding the use of the Nash's isometric embedding.

Introduction

The ordinary Sobolev inequality has been known for many years and its value in the theory of partial differential equations is well known. In [?] Miranda obtained a Sobolev inequality for minimal graphs. A refined version of this new inequality was used by Bombieri, De Giorgi and Miranda to derive gradient bounds for solutions to the minimal surface equation (see [?]).

In [?], a general Sobolev type inequality was presented. That inequality is obtained on what might be termed a generalized manifold and as special cases, results the ordinary Sobolev inequality, a Sobolev inequality on graphs of weak solutions to the mean curvature equation, and a Sobolev inequality on arbitrary C^2 submanifolds of $\mathbb{R}^n$ (of arbitrary co-dimension).

On the other hand, in [?] Allard proves a Sobolev type inequality in a varifold context from a Isoperimetrical inequality for varifolds, for functions with compact support on a varifold V whose first variation δV lies in an appropriate Lebesgue space with respect to $\|\delta V\|$.

In this, joint work with Stefano Nardulli (UFABC), we present a Riemannian intrinsic version of the Allard's result, using a natural extension of the concept of varifold and monotonicity, following the ideas of Simon and Michael in [?] and [?].

Settings

Definition 0.1. *Let (M^n,g) a n-dimensional Riemannian manifold, we define an abstract varifold as a Radon measure on $G_k(M)$, where*

$$G_k(M) := \bigcup_{x\in M}\{x\}\times \mathrm{Gr}(k,T_xM),$$

Let $\mathbf{V}_k(M)$ the space of all k-dimensional varifolds, endowed with the weak topology induced by $C^0_c(G_k(M))$ which is the space of continuous compactly supported functions on $G_k(M^n)$ endowed with the compact open topology.

Definition 0.2. *Let $V \in \mathbf{V}_k(M^n)$, g be a Riemannian metric on M^n. We say that the nonnegative Radon measure on M^n, $||V||$, is the weight of V if $||V|| = \pi_{\#}(V)$. Here, π indicates the natural fiber bundle projection, i.e. for every $A \subseteq G_m(M)$, $x \in M^n$, $S \in G_k(T_xM^n)$, we have $||V||(A) := V(\pi^{-1}(A))$.*

Definition 0.3. *Let (M^n,g) be a n-dimensional Riemannian manifold with Levi-Civita connection ∇, let $\mathfrak{X}^1_c(M)$ be the set of differentiable vector fields on M and $V \in \mathbf{V}_k(M)$ be a k-dimensional varifold $(2\le k\le n)$. We define the first variation of V along the vector field $X \in \mathfrak{X}^1_c(M)$ as*

$$\delta V(X) := \int_{G_k(M)} \operatorname{div}_S X(x)dV(x,S),$$

where

$$\operatorname{div}_S X(x) = \sum_{i=1}^k \langle \tau_i(x), \nabla_{\tau_i}X(x)\rangle_g,$$

and $\{\tau_1(x),\dots,\tau_k(x)\}$ is an orthonormal basis of $S\in \mathrm{Gr}(k,T_xM)$.

Given $V \in \mathbf{V}_k(M)$, using standard approximation theory we see that $\delta V : \mathfrak{X}^0_c(M) \to \mathbb{R}$ defines a linear functional

Proposition 0.1 (First Variation Representation)**.** *Let $V \in \mathbf{V}_k(M)$ such that for all $W \subset\subset U$ open set, there exists a constant $C := C(W)$ such that*

$$|\delta V(X)| \le C\|X\|_{L^\infty(W,V)},$$

for all $X \in \mathfrak{X}^0_c(W)$. Then the total variation $\|\delta V\|$ is a Radon measure on $U \subset M$ open set. Furthermore, there exists H $\|V\|$-measurable function, and $Z\subset M$ with $\|V\|(Z)=0$, such that

$$\delta V(X) = -\int_M \langle X,H\rangle_g\, d\|V\| + \int \langle X,\nu\rangle_g\, d\|\delta V\|_{sing},$$

where ν is a $\|\delta V\|$-measurable function with $\|\nu(x)\|=1$, and $\|\delta V\|_{sing} = \|\delta V\|\llcorner Z$.

In resemble with the manifolds, H is called the generalized mean curvature vector, ν the generalized outer normal and $\|\delta V\|_{sing}$ is called the generalized boundary of V

Monotonicity Formula

We are mainly interested in Varifolds without generalized boundary (i.e $\|\delta V\|_{sing}=0$). Furthermore, since we know than $H \in L^1_{loc}(M, TM : \|V\|)$ then we ask to V to satisfy:

Allard's Conditions:

Let (M^n,g)a Riemannian manifold, such that $Sec_g \le b$, for some $b\in\mathbb{R}$, and $V \in \mathbf{V}_k(M)$. We say that V satisfy **AC** if, for given $X \in \mathfrak{X}^1_c(M)$ such that $\operatorname{spt}\|V\| \subset B_g(\xi,\rho)$, for given $\xi \in M$ and $\rho < inj_{(M,g)}(\xi)$,

$$|\delta V(X)| \le C\left(\int_{B_g(\xi,\rho)} |X|_g^{\frac{p}{p-1}}\, d\|V\|\right)^{\frac{p-1}{p}}.$$

Remark 0.1. *Notice that, by a simple use of Hölder inequality*

$$\textbf{\textit{AC}} \Leftrightarrow \begin{cases} \|\delta V\| \text{ is a Radon measure } \& \ \|\delta V\|_{sing}=0 \\ H \in L^p(M, TM : \|V\|), \end{cases}$$

furthermore

$$\delta V(X) = \int_{G_k(M)} \operatorname{div}_S(X(x))dV(x,S) = -\int_M \langle X(x), H(x)\rangle_g\, d\|V\|(x)$$

Let $u(x) = r_\xi(x) = dist_{(M,g)}(x,\xi)$, and for $\varepsilon>0$ given, let $\gamma \in C^1_c(]-\infty,1[)$ such that

$$\gamma_\varepsilon(y) = 1 \; if\; |y|\le\varepsilon, \quad \gamma_\varepsilon(y) = 0\; if\; |y|>1, \quad \gamma'_\varepsilon(y)<0 \; if\; \varepsilon<|y|<1,$$

and consider,

$$\tilde X_{s,\varepsilon}(x) = \left(\gamma_\varepsilon\left(\frac{u(x)}{s}\right)(u\nabla u)\right)(x), \quad for\; 0<|s|<r_0.$$

Then, for $S\in \mathrm{Gr}(k,T_xM)$

$$\operatorname{div}_S \tilde X_{s,\varepsilon} = \gamma_\varepsilon\left(\frac{u}{s}\right)\operatorname{div}_S(u\nabla u) + \gamma'_\varepsilon\left(\frac{u}{s}\right)\frac{u}{s} - \gamma'_\varepsilon\left(\frac{u}{s}\right)\frac{u}{s}\left|\nabla^{S^\perp}u\right|_g^2.$$

Let $h\in C^1(U)$ a non-negative function, and consider

$$X_{s,\varepsilon}(y) := h(y)\tilde X_s(y) = h(y)\left(\gamma_\varepsilon\left(\frac{u(y)}{s}\right)(u\nabla u)\right)(y), \quad for\; 0<|s|<r_0.$$

Then

$$\operatorname{div}_S X_{s,\varepsilon}(y) = h(y)\operatorname{div}_S \tilde X_{\varepsilon,s} + \left\langle \nabla^S h(y), \tilde X_{\varepsilon,s}\right\rangle_g,$$

therefore,

$$\delta V(X_{s,\varepsilon}) = \int_{G_k(U)} h(y)\gamma_\varepsilon\left(\frac{u(y)}{s}\right)\operatorname{div}_S(u\nabla u)(y)dV(y,S) + \int_{G_k(U)} \frac{h(y)u(y)}{s}\gamma'_\varepsilon\left(\frac{u(y)}{s}\right)dV(y,S)$$
$$- \int_{G_k(U)} \frac{h(y)u(y)}{s}\gamma'_\varepsilon\left(\frac{u(y)}{s}\right)\left|\nabla^\perp u\right|_g^2 + \int_{G_k(U)} \gamma_\varepsilon\left(\frac{u(y)}{s}\right)\langle\nabla^S h(y), (u\nabla u)(y)\rangle_g\, dV(y,S)$$

To compare $\operatorname{div}_S(u\nabla u)$ we make use of the following Lemma, which is an application of the Rauch comparison Theorem.

Lemma 0.1. *Let (M,g) be a complete Riemannian manifold, with Levi-Civita connection ∇, let $b\in\mathbb{R}$ such that $Sec_g\le b$, assume $br_0<\pi$. Then*

$$div_S(u\nabla u)(x) \ge ku(x)\cot_b(u(x)),$$

for all $x\in B_g(\xi,r_0)$, where $\cot_b(s) = cs_b(s)/sn_b(s)$.

In order to apply the Lemma above we have to ask to M to satisfy certain geometrical conditions, that is:

Geometric Conditions:

Let (M^n,g) a Riemannian manifold, we say that M^n satisfy **GC** if, for $\xi\in M$:

1. $Sec_g\le b$ for some $b\in\mathbb{R}$
2. There exists r_0 such that $0<r_0<inj_{(M,g)}(\xi)$ and $r_0b<\pi$.

Theorem 0.2 (Fundamental Weighted Monotonicity Inequality)**.** *Let $(M{\cdot}g)$ a complete Riemannian manifold satisfying **GC** , and $V\in\mathbf{RV}_k(M)$ a varifold satisfying **AC** , then, for all $0<s<r_0$ we have in distributional sense:*

$$\frac{d}{ds}\left(\frac{1}{s^k}\int_{B_g(\xi,s)} h(y)d\|V\|(y)\right) \ge \frac{d}{ds}\int_{B_g(\xi,s)} h\frac{\left|\nabla^\perp u\right|_g^2}{r_\xi^k}d\|V\| + \frac{1}{s^{k+1}}\left(\int_{B_g(\xi,s)}\langle\nabla h + hH, (u\nabla u)\rangle_g\, d\|V\|\right)$$
$$+ c^*\frac{k}{s^k}\int_{B_g(\xi,s)} h(y)d\|V\|(y)$$

where

$$c^* = c^*(r_0,b) := \frac{c(r_0)-1}{s}, \quad \& \quad c := c(s,b) = \begin{cases} s\sqrt{b}\cot\left(\sqrt{b}s\right) & b>0 \\ 0 & b\le 0, \end{cases}$$

Sobolev-Type inequality

Theorem 0.3 (Sobolev Type Inequality)**.** *Let (M^n,g) a complete manifold satisfying **GC** and $V\in\mathbf{RV}_k(M)$ satisfying **AC** , furthermore, assume that for $\xi\in M\cap\operatorname{spt}\|V\|$ given $\Theta^k(x,\|V\|)\ge 1$ for a.e. $x\in B_g(\xi,r_0)$. Let $h\in C^1_c(B_g(\xi,r_0))$ non negative, then there exists $C>0$ such that*

$$\left(\int_M h^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \le C\int_M \left(\left|\nabla^M h\right|_g + h\left(|H|_g - c^*k\right)\right)d\|V\|,$$

Comments on the proof

Notice that the **Fundamental Wighted Monotonicity Inequality** implies that for a.e $\xi\in\operatorname{spt}\|V\|\cap\operatorname{spt}h$

$$\frac{1}{\omega_k\sigma^k}\int_{B_g(\xi,\sigma)} hd\|V\| \le \frac{1}{\omega_k\rho^k}\int_{B_g(\xi,\rho)} hd\|V\| + \int_\sigma^\rho \frac{1}{s^k}\left(\int_{B_g(\xi,s)}\left|\nabla^M h\right|_g + h\left(|H|_g - c^*k\right)d\|V\|\right)ds.$$

Now, we use the following calculus Lemma

Lemma 0.4. *Suppose f and g are bounded non-decreasing functions on $]0,\infty[$, and*

$$1 \le \frac{1}{\sigma^k}f(\sigma) \le \frac{1}{\rho^k}f(\rho) + \int_0^\rho \frac{1}{s^k}g(s)ds, \tag{1}$$

where $0<\sigma<\rho<\infty$. Then, there exists $\rho\in]0,\rho_0[$ such that

$$f(5\rho) \le \frac{1}{2}5^k\rho_0 g(\rho),$$

where $\rho_0 = 2\left(\lim_{\rho\to\infty}f(\rho)\right)^{\frac{1}{n}}$.

Putting together Lemma **??**, the Sobolev Type Inequality and a standard covering argument called the Vitali's 5 Lemma, we get the result.

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