Sobolev Type Inequality for Intrinsic Varifolds

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Abstract
We present a Sobolev type inequality for varifolds intrinsically defined in a natural way, as a consequence of a generalized monotonicity formula for this kind of varifolds and avoiding the use of the Nash’s isometric embedding.

Introduction
The ordinary Sobolev inequality has been known for many years and its value in the theory of partial differential equations is well known. In [7], Miranda obtained a Sobolev inequality for minimal graphs. A refined version of this new inequality was used by Bombieri, De Giorgi and Miranda to derive gradient bounds for solutions to the minimal surface equation (see [7]).

In [7], a general Sobolev type inequality was proved. That inequality is obtained on what might be termed a generalized manifold and as special cases, results the ordinary Sobolev inequality, a Sobolev inequality on graphs of weak solutions to the mean curvature equation, and a Sobolev inequality on arbitrary C^n submanifolds of R^n of arbitrary co-dimension.

On the other hand, in [7] Allard proves a Sobolev type inequality in a varifold context from a homometrical inequality for varifolds, with functions compact support on a varifold whose first variation dV lies in an appropriate Lebesgue space with respect to |dV|.

In this joint work with Stefano Natali (IUFABC), we present a Riemannian intrinsic version of the Allard’s result, using a natural extension of the concept of varifold and monotonicity, following the ideas of Simon and Michael in [7] and [7].

Settings
Definition 0.1. Let (M^n, g) be a n-dimensional Riemannian manifold, we define an abstract varifold as a Radon measure on G^n(M), where

G^n(M) = \{ x \in G(k, T^*_M) | k = \frac{n}{m} \}

Let Μ\((\mu, \nu)\) be the space of all k-dimensional varifolds, endowed with the weak topology induced by C^n(G^n(M)) which is the space of continuous compactly supported functions on G^n(M) endowed with the compact open topology.

Definition 0.2. Let V ∈ Μ\((\mu, \nu)\) be a Riemannian metric on M^n. We say that the nonnegative Radon measure on M^n, ||V||, is the weight of V ∈ Μ\((\mu, \nu)\) if, here, it indicates the natural fiber bundle projection, i.e. for every A ⊆ G^n(M), x ∈ M, \(g \in G(k, T^*_M)\), we have ||V||(A) = ||V||(A(x)).

Definition 0.3. Let (M^n, g) be a n-dimensional Riemannian manifold with Levi-Civita connection \(\nabla\), let X(M^k) be the space of differentiable tensor fields on M and V ∈ Μ\((\mu, \nu)\) be a k-dimensional varifold (k ≤ n ≤ n). We define the first variation of V along the vector field X ∈ X(M^k) as

\[ dV(x) = \int_{\Gamma^x} dV(X)(x, S) \]

where

\[ dV(X)(x) = \sum_{i=1}^{k} \frac{\partial}{\partial x_i} \left( \partial_i V \right) - \left( \xi_{i,x} \right) \cdot \xi_{i,x} \]

and \(\{\xi_{i,x}\} \) is an orthonormal basis of S ∈ G(k, T^*_M).

Given V ∈ Μ\((\mu, \nu)\), using standard approximation we see that H^k(X(M^k)) → R defines a linear functional.

Proposition 0.1 (First Variation Proposition). Let V ∈ Μ\((\mu, \nu)\) such that for all W ⊆ U open set, there exists a constant C = C(V) such that

\[ |dV(X)| \leq C|V|^{|\frac{-1}{n-1}}|W|^{|\frac{n-1}{n}} \]

The total variation ||V|| is a Radon measure on U ⊆ M open set. Furthermore, there holds ||V|| measurable function, and Z ⊆ M with ||V||(Z) = 0, there holds

\[ dV(X) = \int_M (X, \nu_x) - \int_M (X, \nu_x) - dV(x) \]

where \(\nu_x\) is a |V|| measurable function with \(|\nu(x) x| = 1\), and \(|V||\nu_x||x = |V||x.\)

In Riemannian manifolds, \(\nabla\) is called the generalized mean curvature vector, \(\nu\) is the outer normal and \(|V||\nu_x||x\) is called the generalized boundary of V.

Monotonicity Formula
We are mainly interested in Varifolds without generalized boundary (ie \(|\nu||\nu_x||x = 0\)). Furthermore, since we know that H^k(X(M, TM \ | \ V)) then we ask to V to satisfy:

Allard’s Conditions:
Let (M^n, g) a Riemannian manifold, such that \(spt\ V\leq b\), for some b ∈ R, and V ∈ Μ\((\mu, \nu)\). We say that V satisfies AC if, for given X ∈ X(M^k) such that \(\int X = \mu\), we ask \(\nu||V||\leq \mu\) for given ζ ∈ M and \(\nu||V||\leq \mu\) for given ζ ∈ M and \(\nu||V||\leq \mu\).

Remark 0.1. Notice that, by a simple use of Hölder inequality

\[ A C \Leftrightarrow ||V|| \leq \mu \]

furthermore

\[ dV(X) = \int_M (X, \nu_x) - \int_M (X, \nu_x) - dV(x) \]

Let us \(r = r = dist(\nu_x, \partial V)\), and for \(e > 0\) given, let \(\gamma = [\gamma_x(x)] \in (0, \infty)\) such that

\[ \gamma_x(x) = f(e) \]

and consider,

\[ X_{\nu}(x) = \left( \gamma_x(x) + (\nu x)^2 \right)^{\frac{1}{2}} \]

Then, for \(\in G(\mathbb{R}, \mathbb{R})\)

\[ dV(X_{\nu}) = \left( \gamma_x(x) + (\nu x)^2 \right)^{\frac{1}{2}} \]

Let \(h \in C^{1}(\mathbb{R})\) a non-negative function, and consider

\[ \gamma_x(x) = h(y) \]

Then

\[ dV(X_{\nu}) = \left( \gamma_x(x) + (\nu x)^2 \right)^{\frac{1}{2}} = \left( \gamma_x(x) + (\nu x)^2 \right)^{\frac{1}{2}} \]

Therefore, to compare \(dV(X_{\nu})\) we make use of the following Lemma, which is an application of the Rauch comparison Theorem.

Lemma 0.1. Let (M, g) be a complete Riemannian manifold, with Levi-Civita connection \(\nabla\), let \(h \in \mathbb{R}\) such that \(spt\ h\leq b\), assume \(h(z) < 0\). Then

\[ dV(X_{\nu}) \geq dV \]

for all \(\in B(x_0, r_0)\), where \(dV(x_0) = dV(x_0)(\nu x_0(x))\).

In the following chapter we will need to have an upper bound for certain geometrical conditions, that is:

Geometric Conditions:
Let (M^n, g) a Riemannian manifold, we say that M^n satisfies GC if, for ζ ∈ M:

1. \(spt\ V\leq b\) for some b ∈ R
2. There exists \(r_0\) such that \(0 < r_0 < (\nu||V||)(\zeta)\) and \(\nu||V||\leq r_0\).

Theorem 0.2 (Fundamental Weighted Monotonicity Inequality). Let (M^n, g) a complete Riemannian manifold satisfying GC, and \(V \in \mathbb{R}_{+}(\mathbb{R})\) be a varifold satisfying AC, then, for all \(0 < \alpha < \beta\) we have in distributive sense:

\[ \int_M (X, \nu_x) - \int_M (X, \nu_x) - dV(x) \]

where \(\nu\) is a \(|V||\nu_x||x = 1\), and \(|V||\nu_x||x = |V||x.\)

Sobolev-Type Inequality
Theorem 0.3 (Sobolev’s Type Inequality). Let (M^n, g) a complete manifold satisfying GC and \(V \in \mathbb{R}_{+}(\mathbb{R})\) satisfying AC, furthermore, assume that for \(\xi \in M \cap \nu ||V||\) given \(\varepsilon > 0\) we have \(\varepsilon > 0\) for all \(\in B(x_0, r_0)\).

\[ \int_M (X, \nu_x) - \int_M (X, \nu_x) - dV(x) \]

Comments on the proof
Notice that the Fundamental Weighted Monotonicity Inequality implies that for \(\alpha \in \mathbb{R} \cap \mathbb{R}^+\)

\[ \int_M (X, \nu_x) - \int_M (X, \nu_x) - dV(x) \]

Notice that the Fundamental Weighted Monotonicity Inequality implies that for \(\alpha \in \mathbb{R} \cap \mathbb{R}^+\)

\[ 1 \leq \frac{\alpha}{\mu}(\nu||V||) \]

where \(\nu||V||\leq \mu\).

Putting together Lemma 0.1, the Sobolev’s Type Inequality and a standard covering argument called the Vitali’s 5 Lebesgue, we get the result.

References