On certain families of sparse numerical semigroups with Frobenius number even

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Abstract

In this work, we describe and find the genus of certain families of sparse numerical semigroups with Frobenius number even and we also study the realization of the elements on these families as Weierstrass semigroups.

Introduction

Let $\mathbb{Z}$ be the set of integers numbers and $\mathbb{N}_0$ be the set of non-negative integers. A subset $H = \{0 = n_0 < n_1 < \cdots\}$ of $\mathbb{N}_0$ is a *numerical semigroup* if it is closed with respect to addition and its complement $\mathbb{N}_0 \setminus H$ is finite. The cardinality of the set $\mathrm{Gaps}(H) := \mathbb{N}_0 \setminus H$ is called *genus* of the numerical semigroup H and is denoted by $g = g(H)$. Note that $g(H) = 0$ if and only if $H = \mathbb{N}_0$. If $g > 0$, the elements of $\mathrm{Gaps}(H)$ are called *gaps*. The smallest integer c such that $c + h \in H$, for all $h \in \mathbb{N}_0$ is called the *conductor* of H. The least positive integer $n_1 = n_1(H) \in H$ is called the *multiplicity* of H. As $\mathbb{N}_0 \setminus H$ is finite, the set $\mathbb{Z} \setminus H$ has a maximum, which is called *Frobenius number* and is denoted by $\ell_g = \ell_g(H)$. A property known of this number is that $\ell_g \leq 2g - 1$, see [5]. In particular, $H = \mathbb{N}_0$ if and only if -1 is the Frobenius number of H. As a consequence of this fact, we use the notation $\ell_0 := -1$ for all numerical semigroup H. When $g > 0$, we denote $\mathrm{Gaps}(H) = \{1 = \ell_1 < \cdots < \ell_g\}$. So, $c = \ell_g + 1$.

Currently, in literature, there are several special families of numerical semigroups. Examples of these families are the sparse semigroups, which were introduced in [2]. A numerical semigroup H of genus $g > 0$ with $\mathrm{Gaps}(H) = \{\ell_1 < \cdots < \ell_g\}$ is called *sparse numerical semigroup* if $\ell_i - \ell_{i-1} \leq 2$, for all integers i such that $1 \leq i \leq g$. For convenience, we consider the numerical semigroup $\mathbb{N}_0$ as sparse.

For each non-negative integer g, let

$$\mathbb{N}_g := \{0\} \cup \{n \in \mathbb{N} : n \geq g + 1\}$$

(in the case $g = 0$, the notation is itself the $\mathbb{N}_0$). It is clear that $\mathbb{N}_g$ is a numerical semigroup of genus g. The semigroup $\mathbb{N}_g$ is called an *ordinary numerical semigroup* and is a canonical example of sparse numerical semigroups.

This work is organized as follows. In Section 1, we study certain families of sparse numerical semigroups with Frobenius number even. In Section 2, we study the realization of the sparse numerical semigroups determined in the previous section as Weierstrass semigroups.

1 Sparse numerical semigroups with Frobenius number even

For each positive integer k, consider the family $\mathscr{H}_k^{\mathrm{sfe}}$ of sparse numerical semigroups H with genus g and Frobenius number even of the form $2g - 2k$.

If $H \in \mathscr{H}_k^{\mathrm{sfe}}$ in [2, Theorem 2] is proved that $g \leq 6k - 3$ and that the family $\mathscr{H}_k^{\mathrm{sfe}}$ is finite. [7, Question 2.3.10], was conjectured that $g \leq 4k - 1$. This conjecture was proved by Contiero, Moreira and Veloso in [1, Corollary 3.7].

For a numerical semigroup $H = \{0 = n_0 < n_1 < \cdots\}$, we define $M = M(H) := n_1 - 1$. The parameter M was introduced in [2], where if $\ell_g = 2g - 2k$, for some positive integer k, we have that

$$2 \leq M \leq 2k.$$

In this section, we will list the elements of some proper subsets in $\mathscr{H}_k^{\mathrm{sfe}}$ as well as the cardinality of these subsets. For each integer J such that $0 \leq J \leq 2k - 2$, consider the set $\mathscr{H}_{2+J}^k$ of sparse numerical semigroups $H \in \mathscr{H}_k^{\mathrm{sfe}}$ such that $M(H) = 2 + J$. Since

$$\mathscr{H}_k^{\mathrm{sfe}} = \dot{\bigcup_{J=0}^{2k-2}} \mathscr{H}_{2+J}^k,$$

in order to study the cardinality of $\mathscr{H}_k^{\mathrm{sfe}}$ it is enough to study the cardinality of $\mathscr{H}_{2+J}^k$, for all integers J such that $0 \leq J \leq 2k - 2$. Next we will study the cases: $J = 2k - 2$, $J = 2k - 3$ and $J = 2k - 4$.

Theorem 1 ($J = 2k - 2$)**.** *For all integers $k \geq 1$,*

$$\mathscr{H}_{2k}^k = \{\mathbb{N}_{2k}\} \,\dot\cup\, \left\{H_{(k,r)} : r \in \mathbb{N},\ 1 \leq r \leq k\right\},$$

where

$$H_{(k,r)} = \{2k + 2i - 1 : i \in \mathbb{N},\ 1 \leq i \leq r\} \,\dot\cup\, \mathbb{N}_{2k+2r},$$

for $1 \leq r \leq k$. In addition, $g\left(H_{(k,r)}\right) = 2k + r$. In particular, $\left|\mathscr{H}_{2k}^k\right| = k + 1$.

Theorem 2 ($J = 2k - 3$)**.** *For all integers $k \geq 2$,*

$$\mathscr{H}_{2k-1}^k = \left\{H_{(k,r)}^{\alpha_r} : (r, \alpha_r) \in \mathbb{N}^2,\ 1 \leq r \leq k - 1,\ 1 \leq \alpha_r \leq r\right\},$$

where

$$H_{(k,r)}^{\alpha_r} = \{2(k + \mu - 1) : 1 \leq \mu \leq \alpha_r\} \,\dot\cup\, \{2k + 2\nu + 1 : \alpha_r \leq \nu \leq r - 1\} \,\dot\cup\, \mathbb{N}_{2k+2r},$$

for $1 \leq r \leq k - 1$ and $1 \leq \alpha_r \leq r$. In addition, $g\left(H_{(k,r)}^{\alpha_r}\right) = 2k + r$. In particular, $\left|\mathscr{H}_{2k-1}^k\right| = \binom{k}{2}$.

Theorem 3 ($J = 2k - 4$)**.** *For all integers $k \geq 3$,*

$$\mathscr{H}_{2k-2}^k = \left\{H_{(k,r)}^{(s,\alpha_s)} : (r, s, \alpha_s) \in \mathbb{N}^3, 1 \leq r \leq k - 2, 1 \leq s \leq r, 1 \leq \alpha_s \leq r - s + 1\right\} \,\dot\cup\, \left\{H_{(k,r)}^{(1,k)} : r \in \{k, k + 1\}\right\},$$

where

$$H_{(k,r)}^{(s,\alpha_s)} = \{2k + 2\lambda - 3 : 1 \leq \lambda \leq s\} \,\dot\cup\, \{2(k + s + \mu - 1) : 1 \leq \mu \leq \alpha_s - 1\} \,\dot\cup\, \{2k + 2\nu + 2s - 1 : \alpha_s \leq \nu \leq r - s\} \,\dot\cup\, \mathbb{N}_{2k+2r},$$

for $1 \leq r \leq k - 2$, $1 \leq s \leq r$ and $1 \leq \alpha_s \leq r - s + 1$; for $r \in \{k, k + 1\}$, let $s = 1$ and $\alpha_s = k$. In addition, $g\left(H_{(k,r)}^{(s,\alpha_s)}\right) = 2k + r$. In particular, $\left|\mathscr{H}_{2k-2}^k\right| = \binom{k}{3} + 2$.

2 On Weierstrass semigroup

Let $\mathcal{X}$ be a non-singular, projective, irreducible, algebraic curve of genus $g \geq 1$ over a field $\mathbb{K}$. Let $\mathbb{K}(\mathcal{X})$ be the field of rational functions on $\mathcal{X}$ and for $f \in \mathbb{K}(\mathcal{X})$, $(f)_\infty$ will denote the divisor of poles of f. Let P be a point on $\mathcal{X}$. The set

$$H(P) := \{n \in \mathbb{N}_0 : \text{there exist } f \in \mathbb{K}(\mathcal{X}) \text{ with } (f)_\infty = nP\},$$

is a numerical semigroup called *Weierstrass semigroup* of $\mathcal{X}$ at P.

Given a numerical semigroup H, is there a curve $\mathcal{X}$ such that for some point $P \in \mathcal{X}$ has $H = H(P)$? If the answer is yes, we say that the numerical semigroup H is *Weierstrass*.

In addition to the genus $g(H)$, Frobenius number $\ell_g(H)$ and multiplicity $n_1(H)$, an important concept in this study is the *weight* of a numerical semigroup H. If $\mathrm{Gaps}(H) = \{1 = \ell_1 < \cdots < \ell_g\}$ be the gaps set of H, the *weight* of H is

$$w(H) = \sum_{i=1}^{g} (\ell_i - i).$$

As a particular result, a numerical semigroup H is Weierstrass if the following condition holds:

$$\begin{gathered}\text{either } w(H) \leq g(H)/2, \text{ or} \\ g(H)/2 < w(H) \leq g(H) - 1 \text{ and } 2n_1(H) > \ell_g(H)\end{gathered} \qquad (*)$$

(see Eisenbud-Harris [3], Komeda [4]).

Next, we will see which of the semigroups in the families studied in the previous section are Weierstrass.

Lemma 1. *Let $k \geq 2$ be an integer. If $H = H_{(k,r)}^{\alpha_r} \in \mathscr{H}_{2k-1}^k$, then $w(H) = \alpha_r + \frac{r(r+1)}{2}$ and $2n_1(H) > \ell_g(H)$.*

Proposition 1. *Let $k \geq 2$ be an integer and $H = H_{(k,r)}^{\alpha_r} \in \mathscr{H}_{2k-1}^k$. If $\alpha_r + \frac{r(r-1)}{2} \leq 2k - 1$, then H is Weierstrass. In particular, if $k \geq 3$ and $r \in \left\{1, \ldots, \left\lfloor \frac{-1+\sqrt{16k-7}}{2} \right\rfloor\right\}$, then H is Weierstrass.*

Proof. By previous lemma, $2n_1(H) > \ell_g(H)$. So, from the condition $(*)$ above, follows that H is Weierstrass if $w(H) \leq g(H) - 1$. On the other hand, since $g(H) = 2k + r$, and, by Lemma 1, $w(H) = \alpha_r + \frac{r(r+1)}{2}$. Thus, H is Weierstrass if $\alpha_r + \frac{r(r+1)}{2} \leq 2k + r - 1$, that is, if $\alpha_r + \frac{r(r-1)}{2} \leq 2k - 1$.

Now, if $r \in \left\{1, \ldots, \left\lfloor \frac{-1+\sqrt{16k-7}}{2} \right\rfloor\right\}$, then $r + \frac{r(r-1)}{2} \leq 2k - 1$. Also, since $k \geq 3$, we have that $r \leq k - 1$. So, since $\alpha_r \leq r$, the required result follows. □

Lemma 2. *Let $k \geq 3$ be an integer. If $H = H_{(k,r)}^{(s,\alpha_s)} \in \mathscr{H}_{2k-2}^k$, then $w(H) = 2s - 1 + \alpha_s + \frac{r(r+1)}{2}$. In addition, if $r \notin \{k, k + 1\}$, then $2n_1(H) > \ell_g(H)$.*

Proposition 2. *Let $k \geq 3$ be an integer and $H = H_{(k,r)}^{(s,\alpha_s)} \in \mathscr{H}_{2k-2}^k$. If $r \notin \{k, k + 1\}$ and $2s + \alpha_s + \frac{r(r-1)}{2} \leq 2k$, then H is Weierstrass. In particular, if $k \geq 4$ and $r \in \left\{1, \ldots, \left\lfloor \frac{-3+\sqrt{16k+1}}{2} \right\rfloor\right\}$, then H is Weierstrass.*

Proof. The proof of the first statement is analogous to the proof of the Proposition 1.

Now, if $r \in \left\{1, \ldots, \left\lfloor \frac{-3+\sqrt{16k+1}}{2} \right\rfloor\right\}$, then $2r + \frac{r(r-1)}{2} \leq 2k - 1$. Also, since $k \geq 4$, we have that $r \leq k - 2$. So, since $s \leq r$ and $\alpha_s \leq r - s + 1$, the required result follows. □

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