Some Aspects to de Korteweg - de Vries Equation

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Resumo

The Korteweg- de Vries [KdV] equation is a mathematical model of waves on shallow water surfaces. It is particularly notable as the prototypical example of an exactly solvable model, that is, a non-linear partial differential equation whose solution can be exactly and precisely specified. KdV can be solved by means of the inverse scattering transform. The mathematical theory behind the KdV equation is a topic of active research. The KdV equation was first introduced by Boussinesq (1877) and rediscovered by **Diederik Korteweg** and **Gustav de Vries** (1895).

Introduction

The Korteweg - de Vries equation forms one of a group of equations governing nonlinear dispersive wave motion which are solvable on the infinite line by the technique of inverse scattering. In this work, we want to present the solution of the initial value problem for the kdv equation, described as:

$$\frac{\partial u}{\partial t} + au\frac{\partial u}{\partial x} + b\frac{\partial^3 u}{\partial t^3} = 0 \tag{1}$$

Where a and b represent the nonlinearity and the dispersion respectively. The original derivation of Korteweg and de Vries brings these parameters in terms of physical quantities such as acceleration of gravity, channel depth, etc. Here, for simplicity, we will keep a and b only as generic parameters. Then let's assume the constants $a = 6$ and $b = 1$, The factor 6 is just a scaling factor to make solutions easier to describe

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial t^3} = 0 \tag{2}$$

Study for Exact Solution to the KdV Equation

In this section we shall examine solutions of the KdV equation [2] and the methods one can use for obtaining them. We first start by looking for travelling wave solutions. For that we make the Ansatz $u(x,t) = z(x - ct) \equiv z(\zeta)$ for some $c \in \Re$. By inserting this into the KdV equation yields an ordinary differential equation for z, namely $-c\frac{dz}{d\zeta} + 6z\frac{dz}{d\zeta} + \frac{d^3z}{d\zeta^3} = 0$, which can be easily integrated yielding $-cz + 3z^2 + \frac{d^2z}{d\zeta^2} = c_1$ where c_1 is an integration constant. We then multiply the latter equation by $\frac{dz}{d\zeta}$, integrate it again, thus obtaining an additional integration constant c_2. By rescaling the (integration) constants, we find:

$$(\frac{dz}{d\zeta})^2 = -2z^3 + cz^2 + c_1z + c_2 \equiv F(z)$$

We have thus obtained a first order non-linear (ordinary) differential equation which needs to be solved. Obviously the roots of F will play an important role in our analysis and we can distinguish the following cases

1. F has three distinct (real) roots.
2. F has two (real) roots, one has order two.
3. F has one simple root (and two imaginary ones).
4. F has one root of order three.

For details on the F function for analytical solutions, see [5] making some conditions for the function F and its derivatives we have in case $x \pm \infty$ we should have $z \to 0, \frac{dz}{d\zeta} \to 0, \frac{d^2z}{d\zeta^2} \to 0$. From these requirements it follows $c_1 = c_2 = 0$.[1] **Case Particular** $c_1 = c_2 = 0$ In this case, we will have that the equation $\left(\frac{dz}{d\zeta}\right)^2 = -2z^3 + cz^2 + c_1z + c_2$ is reduced to:

$$\frac{dz}{d\zeta} = \sqrt{-2z^3 + cz^2} \tag{3}$$

By seperation of variables we may write

$$\frac{dz}{z\sqrt{c - 2z}} = d\zeta \tag{4}$$

through some algebraic manipulations and using appropriate transformations we obtain

$$z(\zeta) = \frac{c}{2}\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}\zeta\right) \tag{5}$$

Now we use $\zeta = x - ct$ and we finally get

$$u(x - ct) = \frac{c}{2}\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(x - ct)\right) \tag{6}$$

In order to have a real solution the quantity c must be a positive number. As it is easily seen from [?] for $c > 0$ the solitary wave moves to the right. The second point is that the amplitude is proportional to the speed which is indicated by the value of . Thus larger amplitude solitary waves move with a higher speed than smaller amplitude waves. The figure, shows the propagation of such a wave for different times .

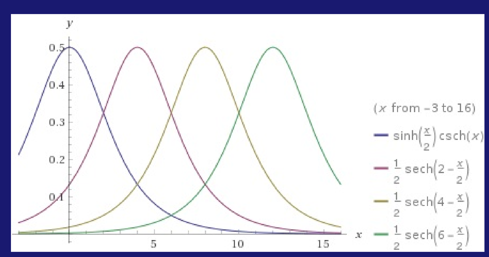

Figura 1: Solitary wave at $t = 0, t = 4, t = 8$ and $t = 12$

Work in Progress for the Study of KdV

Solving a nonlinear equation is somewhat cumbersome, so we present the Hirota method for the construction of multisolitonic solutions for non-linear integrable systems. Multisolitonic solutions can, of course, be derived by other methods, such as reverse spreading, for example. The advantage of the Hirota method is that it is more algebraic than analytical, in addition to being faster to produce results. Let's discuss it in a bit more detail in the context of the KdV equation.

Referências

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[1] More general solutions can be found for other choices of c and . These solutions can be c_1, c_2 represented in terms of elliptic integrals, for details see [?]