Existence and uniqueness of weak solutions for the non-autonomous Ladyzhenskaya model and their asymptotic pullback behavior in L^2

Heraclio López[1] & Gabriela Planas[1] & Pedro Marin-Rubio[2]

[1]University of Campinas, [2]Universidad de Sevilla

heracliolopezl@gmail.com

Abstract

We establish a result of existence of pullback attractors in H for the Ladyzhenskaya model of incompressible viscous fluid in a domain $\Omega \subset \mathbb{R}^n$, $n \in \{2,3\}$. The motion of incompressible viscous fluid in Ω, characterized by the velocity field $u = (u_1, ..., u_n)$ and the pressure π, is governed by a system of $n+1$ equations.

Mathematical Problem

$$(LM)\begin{cases} \frac{\partial u}{\partial t} - div_x S(Du) + div_x(u \otimes u) + \nabla_x \pi = f(t) \\ \qquad\qquad\qquad\qquad in\ (\tau, +\infty) \times \Omega, \\ div_x u = 0 \quad in\ (\tau, +\infty) \times \Omega, \\ u(\tau, x) = u_\tau(x), \quad x \in \Omega, \\ u = 0 \quad on\ (\tau, +\infty) \times \partial\Omega, \end{cases}$$

We will assume that $\mathbb{S}$ is a potential, meaning there exists a function $\Phi : \mathbb{R}^{n\times n} \to \mathbb{R}$ of class C^2 such that

$$\partial_D \Phi(D) = \mathbb{S}(D),\ \partial_D^2\Phi(D) : (B \otimes B) \geq \nu_1(1 + \mu|D|)^{p-2}|B|^2,$$
$$|\partial_D^2 \Phi(D)| \leq c_4\nu_1(1 + \mu|D|)^{p-2}.$$

Functional spaces:

$$\begin{aligned} &\mathcal{V} := \{\varphi \in C_c^\infty(\Omega; \mathbb{R}^n) \mid div_x \varphi = 0\}; \\ &H = \overline{\mathcal{V}}^{(L^2)^n} \quad and \quad V_p = \overline{\mathcal{V}}^{(W^{1,p})^n}. \end{aligned} \tag{1}$$

Definition 1: By a weak solution to (LM) we understand a function u, belonging to the class

$$u \in L^p(\tau, T; V_p) \cap L^\infty(\tau, T; H)$$

such that, for all $v \in V_p$, satisfies the identity

$$\frac{d}{dt}(u(t), v) + \langle T(u(t)), v\rangle + \langle B(u(t)), v\rangle = \langle f(t), v\rangle,$$

with

$$u(\tau) = u^\tau.$$

Existence and Uniqueness

Theorem 1: (Existence) Let us consider τ, T $(T > \tau)$, $u_\tau \in H$, $f \in (L^p(\tau, T; V_p))^*$. If $p \geq 1 + \frac{2n}{n+2}$, then there exists at least one weak solution of the problem (LM).

Theorem 2: (Uniqueness)

1. If $n = 2$, the weak solutions are unique.
2. If $n = 3$, any weak solution that has additional regularity $u \in L^{\frac{2p}{2p-3}}(\tau, T; V_p)$ is unique in the class of weak solutions.

Existence of Pullback Attractor in H

Let $\mathcal{R}$ be the set of all functions $\rho : \mathbb{R} \to [0, \infty)$ such that

$$e^{\bar{\eta}\tau}\rho^2(\tau) \to 0 \quad as\ \tau \to -\infty,$$

where $\bar{\eta} = \frac{2c_2\nu_1\lambda_1}{c_0^2}$.

Definition 2:(Universe). We will denote by $\mathcal{D}^H$ the class of all families $\widehat{D} := \{D(t) : t \in \mathbb{R},\ D(t) \subset H,\ D(t) \neq \emptyset\}$, such that $D(t) \subset \{u \in H : |u|_2 \leq \rho_{\widehat{D}}(t)\}$ for some $\rho_{\widehat{D}} \in \mathcal{R}$.

We assume that $f \in L^{p'}_{loc}(\mathbb{R}; V_p^*)$ and satisfies

$$\int_{-\infty}^0 e^{\bar{\eta}s}\|f(s)\|_*^{p'} ds < \infty. \tag{2}$$

Lemma 1: Under the assumptions of Theorem 1. Then for any $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}^H$, there exists $\tau_1(\widehat{D}, t) < t - 2$, such that $\tau \leq \tau_1(\widehat{D}, t)$ and for any $u_\tau \in D(\tau)$, it holds

$$\begin{cases} |u(r; \tau, u_\tau)|_2 \leq R_1(t) \quad\ for\ all\ r \in [t-2, t], \\ \int_{r-1}^r |\nabla u(s; \tau, u_\tau)|_2^2 ds \leq R_2(t) \quad\ for\ all\ r \in [t-1, t], \\ \int_{r-1}^r \|\nabla u(s; \tau, u_\tau)\|_p^p ds \leq R_3(t) \quad\ for\ all\ r \in [t-1, t], \end{cases}$$

where

$$R_1^2(t) = 1 + c_{\nu_2, p, p'}\int_{-\infty}^t e^{-\bar{\eta}(t-s)}\|f(s)\|_*^{p'} ds,$$

$$R_2(t) = \frac{c_0^2}{2c_2\nu_1}R_1^2(t) + \frac{c_0^2}{c_2\nu_1 p'\epsilon^{p'}}\int_{t-2}^t \|f(s)\|_*^{p'} ds,$$

$$R_3(t) = \frac{\tilde{c}_0^p}{c_2\nu_2}R_1^2(t) + \frac{2\tilde{c}_0^p}{c_2\nu_2 p'\epsilon^{p'}}\int_{t-2}^t \|f(s)\|_*^{p'} ds.$$

Corollary 1: The family of sets

$$\widehat{D}_0 = \{\{u \in H : |u|_2 \leq R_1(t)\} : t \in \mathbb{R}\},$$

is pullback $\mathcal{D}^H$-absorbing family for the solution process $S(\cdot,\cdot)$, that is, for any $\widehat{D} = \{D(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}^H$ and any $t \in \mathbb{R}$, there is a $\tau_1(t, \widehat{D}) < t - 2$ satisfying

$$S(t, \tau)D(\tau) \subset \{u \in H : |u|_2 \leq R_1(t)\} \qquad for\ all\ \tau \leq \tau_1(t, \widehat{D}).$$

Moreover, $\widehat{D}_0$ belongs to $\mathcal{D}^H$.

Lemma 2: The process $S(\cdot,\cdot)$ is pullback $\mathcal{D}^H$-asymptotically compact.

Theorem 3: Under the assumptions of Theorems 1 and 2. Then the family $\mathcal{A}_{\mathcal{D}^H} = \{\mathcal{A}_{\mathcal{D}^H}(t) : t \in \mathbb{R}\}$ defined by

$$\mathcal{A}_{\mathcal{D}^H}(t) = \Lambda(\widehat{D}_0, t), \quad t \in \mathbb{R},$$

where $\widehat{D}_0$ is given by (), and for any $\widehat{D} \in \mathcal{D}^H$,

$$\Lambda(\widehat{D}, t) := \bigcap_{\sigma \leq t}\overline{\left(\bigcup_{\tau \leq \sigma} S(t, \tau)D(\tau)\right)}^H \quad \forall t \in \mathbb{R}$$

is the unique pullback $\mathcal{D}^H$-attractor for the process $S(\cdot,\cdot)$ belonging to $\mathcal{D}^H$. in addition, $\mathcal{A}_{\mathcal{D}^H}$ satisfies

$$\mathcal{A}_{\mathcal{D}^H}(t) = \overline{\bigcup_{\widehat{D} \in \mathcal{D}^H} \Lambda(\widehat{D}, t)}^H \quad \forall t \in \mathbb{R}.$$

Furthermore, $\mathcal{A}_{\mathcal{D}^H}$ is minimal.

Referências

[1] J. A. Carvalho, A. N.; Langa and J. C. Robinson. *Attractors for infinite-dimensional non-autonomous dynamical systems*. Applied Mathematical Sciences, Springer-Verlag, 2012.

[2] E. Feireisl and D. Prazák. *Asymptotic Behavior of Dynamical systems in fluid Mechanics*. AIMS Series on Applied Mathematics, 2010.

[3] J. L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Dunod, Paris, 1969.

[4] J. Málek and J. Nečas. A finite-dimensional attrator for three-dimensional flow of incompressible fluids. *Journal of Differential Equations*, 127:498–518, 1996.

[5] J. Málek and D. Prazḱ. Finite fractal dimension of the global attractor for a class of non-newtonian fluids. *Applied Mathematics Letters*, 13:105–110, 2000.

[6] J.; Rokyta M. Málek, J.; Nečas and Ružička. M. *Weak and Measure-Valued Solutions to Evolutionary PDEs*. London: Chapman & Hall, 1996.

Agradecimentos

To CNPq and CAPES by the financial support.

Gabriela Planas[1] & Pedro Marin-Rubio[2]

[1]Universidade Estadual de Campinas [2]Universidad de Sevilla