# Remarks on V-static spaces on compact manifolds with boundary


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### Abstract

The aim of this poster is to provide a general Bochner type formula which enables us to prove some rigidity results for V-static spaces. In particular, we show that an n-dimensional positive static triple with connected boundary and positive scalar curvature must be isometric to the standard hemisphere, provided that the metric has zero radial Weyl curvature and satisfies a suitable pinching condition. Moreover, we classify V-static spaces with nonnegative sectional curvature. This result is contained in [2].


### Introduction

Let $(M^n,\, g)$ be a connected Riemannian manifold. We say that $g$ is a $V$*-static metric* if there is a smooth function $f$ on $M^n$ and a constant $\kappa$ satisfying the $V$-static equation

$$\mathfrak{L}_g^*(f) = -(\Delta f)g + Hess\, f - fRic = \kappa g, \qquad (1)$$

where $\mathfrak{L}_g^*$ is the formal $L^2$-adjoint of the linearization of the scalar curvature operator $\mathfrak{L}_g$. Such a function $f$ is called $V$*-static potential*.

In the work [1] and [3], Ambrozio and Batista et al., respectively, obtained interesting classification results for V-static three-dimensional compact manifolds. To do so, they proved a Böchner type formula for such structures involving the traceless Ricci tensor and the Cotton tensor. Those formulae may be used to rule out some possible new examples. In this work, we extend such Böchner type formulae for a more general class of metrics and arbitrary dimension $n > 2$. More precisely, we have established the following result.

**Theorem 1.** *Let $(M^n,\, g,\, f, \kappa)$ be a connected, smooth Riemannian manifold and $f$ is a smooth function on $M^n$ satisfying the $V$-static equation (1). Then we have:*

$$\begin{aligned}\frac{1}{2}\mathrm{div}(f\nabla|Ric|^2) &= \Big(\frac{n-2}{n-1}|C_{ijk}|^2 + |\nabla Ric|^2\Big)f + \frac{n\kappa}{n-1}|\mathring{Ric}|^2 \\ &+\Big(\frac{2}{n-1}R|\mathring{Ric}|^2 + \frac{2n}{n-2}tr(\mathring{Ric}^3)\Big)f \\ &-\frac{n-2}{n-1}W_{ijkl}\nabla_l f C_{ijk} - 2fW_{ijkl}R_{ik}R_{jl},\end{aligned}$$

*where $C$ stands for the Cotton tensor, $W$ is the Weyl tensor and $\mathring{Ric}$ is the traceless Ricci tensor.*

### Positive static triple

We remember that a *positive static triple* is a triple $(M^n,\, g,\, f)$ consisting of a connected $n$-dimensional smooth manifold $M$ with boundary $\partial M$ (possibly empty), a complete Riemannian metric $g$ on $M$ and a nontrivial static potential $f \in C^\infty(M)$ that is non-negative, vanishes precisely on $\partial M$ and satisfies the static equation $\mathfrak{L}_g^*(f) = 0$.

The first application of the Böchner type formula will be used to give a positive answer for *cosmic no-hair conjecture* which was formulated by Boucher, Gibbons and Horowitz in 1984. Such conjecture assert that the only static vacuum spacetime with positive cosmological constant and connected event horizon is the de Sitter space of radius $r$. To be precise, we have established the following results.

**Theorem 2.** *Let $(M^n,\, g,\, f)$ be a compact, oriented, connected positive static triple with positive scalar curvature. Suppose that:*

- *$M^n$ has zero radial Weyl curvature and*
- *$|\mathring{Ric}|^2 \leq \frac{R^2}{n(n-1)}$.*

*Then one of the following assertions holds:*

1. *$M^n$ is equivalent to the standard hemisphere of $\mathbb{S}^n$; or*
2. *$|\mathring{Ric}|^2 = \frac{R^2}{n(n-1)}$ and $(M^n,\, g,\, f)$ is covered by a static triple that is equivalent to the standard cylinder.*

**Theorem 3.** *Let $(M^n,\, g,\, f)$ be a positive static triple with nonnegative sectional curvature, zero radial Weyl curvature and scalar curvature $R = n(n-1)$ Then Then, up to a finite quotient, $M^n$ is isometric to either the standard hemisphere $\mathbb{S}^n_+$ or the standard cylinder over $\mathbb{S}^n$.*

### Miao-Tam critical metrics

The case where $\kappa \neq 0$ and the potential function $f$ vanishes on the boundary was studied by Miao and Tam. In this approach, a *Miao-Tam critical metric* is a 3-tuple $(M^n,\, g,\, f)$, where $(M^n,\, g)$ is a compact Riemannian manifold of dimension at least three with a smooth boundary $\partial M$ and $f : M^n \to \mathbb{R}$ is a smooth function such that $f^{-1}(0) = \partial M$ satisfying $\mathfrak{L}_g^*(f) = g$.

In 2011, Miao and Tam posed the question of whether there exist non-constant sectional curvature Miao-Tam critical metrics on a compact manifold whose bound- ary is isometric to a standard round sphere. In order to give a partial answer to this problem, they showed that an Einstein Miao-Tam critical metric must be isometric to a geodesic ball in a simply connected space form $\mathbb{R}^n$, $\mathbb{H}^n$ or $\mathbb{S}^n$. In this special case the restriction on the boundary was not necessary. Here, we shall give a partial answer for this problem assuming that the metric has zero radial Weyl curvature and satisfies a suitable pinching condition.

**Theorem 4.** *Let $(M^n,\, g,\, f)$ be a compact, oriented, connected Miao-Tam critical metric with positive scalar curvature and nonnegative potential function $f$. Suppose that:*

- *$M^n$ has zero radial Weyl curvature and*
- *$|\mathring{Ric}|^2 \leq \frac{R^2}{n(n-1)}$.*

*Then $M^n$ must be isometric to a geodesic ball in $\mathbb{S}^n$.*

Moreover, we classify Miao-Tam metrics with nonnegative sectional curvature. In fact, we get the following rigidity result.

**Theorem 5.** *Let $(M^n,\, g,\, f)$ be a Miao-Tam critical metric with with nonnegative sectional curvature and zero radial Weyl curvature, $f$ is also assumed to be nonnegative. Then $M^n$ is isometric to a geodesic ball in a simply connected space form $\mathbb{R}^n$ or $\mathbb{S}^n$.*

### Referências


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