Pseudo-parallel surfaces in $\mathbb{Q}_c^n \times \mathbb{R}$

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Abstract

In this work we give a characterization of pseudo-parallel surfaces in $\mathbb{S}_c^n \times \mathbb{R}$ and $\mathbb{H}_c^n \times \mathbb{R}$, extending an analogous result by Asperti-Lobos-Mercuri for the pseudo-parallel case in space forms. Moreover, when $n = 3$, we prove that any pseudo-parallel surface has flat normal bundle. We also give examples of pseudo-parallel surfaces which are neither semi-parallel nor pseudo-parallel surfaces in a slice. Finally, when $n \geq 4$ we give examples of pseudo-parallel surfaces with non vanishing normal curvature.

Preliminaries

$\mathbb{Q}_c^n$ with $c \neq 0$ to refer the sphere n-space $\mathbb{S}_c^n$ or the hyperbolic n-space $\mathbb{H}_c^n$.
An isometric immersion $f : M^m \to \mathbb{Q}_c^n \times \mathbb{R}$ is said to be:

- (i) *totally geodesic if* $\alpha = 0$;
- (ii) *parallel if* $(\nabla_X \alpha) = 0$;
- (iii) *semi-parallel if* $\tilde{R}(X, Y) \cdot \alpha = 0$;
- (iv) *pseudo-parallel if* $\tilde{R}(X, Y) \cdot \alpha = \Phi X \wedge Y \cdot \alpha$,

for some smooth function Φ in M^m and any vector fields X, Y in M^m. Here, α denotes the second fundamental form of f and $\tilde{R} = R \oplus R^\perp$ denotes the curvature tensor of $\mathbb{Q}_c^n \times \mathbb{R}$. The concept of pseudo-parallel immersions was first introduced by Asperti-Lobos-Mercuri in [1] as a generalization of semi-parallel immersions. Also in [1], authors investigated pseudo-parallel surfaces in space forms. They obtained the following result:

Theorem (Asperti-Lobos-Mercuri [1])

Let $f : M^2 \to \mathbb{Q}_c^4$ be a surface with $R^\perp \neq 0$. Then f is pseudo-parallel if and only if f is superminimal, that is, f is a minimal immersion and is λ-isotropic.

Also, they classified such surfaces of codimension 3 and codimension 4 with constant pseudo-parallelism function.
We recall that an isometric immersion $f : M^n \to \tilde{M}^m$ is said to be λ-*isotropic* if $\|\alpha^f(X, X)\| = \lambda(p), \quad \forall X \in T_pM, \quad \forall p \in M^n$ with $\|X\| = 1$.
On the other hand, M. Sakaki studied surfaces in $\mathbb{S}^3 \times \mathbb{R}$ and $\mathbb{H}^3 \times \mathbb{R}$, showing in [3] the following theorem:

Theorem (Sakaki [3])

Let $f : M^2 \to \mathbb{Q}_c^3 \times \mathbb{R}$ a minimal surface with $c \neq 0$. If f is λ-isotropic at any point, then f is a totally geodesic immersion.

By the Fundamental Equations and pseudo-parallelism condition we get the relations:

$$R^\perp(e_1, e_2)\alpha_{11} = 2(\Phi - K)\alpha_{12}, = -R^\perp(e_1, e_2)\alpha_{22} \quad (1)$$
$$R^\perp(e_1, e_2)\alpha_{12} = (K - \Phi)(\alpha_{11} - \alpha_{22}), \quad (2)$$
$$K = c(1 - \|T\|^2) + \langle \alpha_{11}, \alpha_{22} \rangle - \|\alpha_{12}\|^2, \quad (3)$$

where $\{e_1, e_2\}$ is an orthonormal frame of M^2, $\alpha_{ij} = \alpha(e_i, e_j)$, K is the Gaussian curvature of M^2 and is the tangent part of $\frac{\partial}{\partial t}$, the canonical unit vector field tangent to the second factor of $\mathbb{Q}_c^n \times \mathbb{R}$.

Proposition 1

Let $f : M^2 \to \mathbb{Q}_c^n \times \mathbb{R}$ be a surface with flat normal bundle. Then f is pseudo-parallel immersion.

Proof

Since f has flat normal bundle, by equations (1) to (2) we conclude that f is Φ-pseudo-parallel by taking $\Phi = K$, where K is the Gaussian curvature of M^2.

We have two propositions that is useful to construct examples of pseudo-parallel surfaces.

Proposition 2

Let $f : M^m \to \mathbb{Q}_c^n$ be an isometric immersion and let $j : \mathbb{Q}_c^n \to \mathbb{Q}_c^n \times \mathbb{R}$ be a totally geodesic immersion. If f is Φ-pseudo-parallel, then $j \circ f$ is Φ-pseudo-parallel.

Proposition 3

Let $f : M^m \to \mathbb{Q}_c^n \times \mathbb{R}$ be an isometric immersion and let $j : \mathbb{Q}_c^n \times \mathbb{R} \to \mathbb{Q}_c^{n+l} \times \mathbb{R}$ be a totally geodesic immersion. If f is Φ-pseudo-parallel, then $j \circ f$ is Φ-pseudo-parallel.

The Result: A theorem of characterization

Let $f : M^2 \to \mathbb{Q}_c^n \times \mathbb{R}$ be a pseudo-parallel surface which does not have flat normal bundle on any open subset of M^2. Then $n \geq 4$, f is λ-isotropic and

$$K > \Phi, \quad (4)$$
$$\lambda^2 = 4K - 3\Phi + c(\|T\|^2 - 1) > 0, \quad (5)$$
$$\|H\|^2 = 3K - 2\Phi + c(\|T\|^2 - 1) \geq 0, \quad (6)$$

where K is the Gaussian curvature, λ is a smooth real-valued function on M^2, H is the mean curvature vector field of f and T is the tangent part $\frac{\partial}{\partial t}$, the canonic unit vector field tangent to the second factor of $\mathbb{Q}_c^n \times \mathbb{R}$.
Conversely, if f is λ-isotropic then f is pseudo-parallel.

Remark

Theorem A extends for $\mathbb{Q}_c^n \times \mathbb{R}$ a similar result of pseudo-paralell surfaces into space forms given by Asperti-Lobos-Mercuri in [1].

Some examples

For the parametrizations $f_i : \mathbb{R}^2 \to \mathbb{Q}_c^3 \times \mathbb{R}$ below, we consider $0 < d < 1, k > 0, a \neq 0$ and $b \in \mathbb{R}$. The first example is a semi-parallel surface in $\mathbb{S}_c^3 \times \mathbb{R}$ which is not parallel. The second and third are pseudo-parallel surfaces in $\mathbb{S}_c^3 \times \mathbb{R}$ and $\mathbb{H}_c^3 \times \mathbb{R}$, respectively, and both are not semi-parallel. In all the cases $0 < \|T\| < 1$, that is, f is not just an inclusion of a pseudo-parallel surface in $\mathbb{Q}_c^3$ into $\mathbb{Q}_c^3 \times \mathbb{R}$.

$$f_1(u, v) = \frac{1}{\sqrt{c}}(\sqrt{1 - d^2}\cos\theta(u), \sqrt{1 - d^2}\sin\theta(u), d\cos v, d\sin v, kv),$$
$$f_2(u, v) = \frac{1}{\sqrt{c}}(d\cos u, d\sin u\cos v, d\sin u\sin v, \sqrt{1 - d^2}, au + b),$$
$$f_3(u, v) = \frac{1}{\sqrt{-c}}(d\cosh u, d\sinh u\cos v, d\sinh u\sin v, \sqrt{d^2 - 1}, au + b).$$

Let $f : \mathbb{R}^2 \to \mathbb{S}_c^5$ be the surface given by (see [2])

$$f(x, y) = \frac{2}{\sqrt{6c}}(\cos u\cos v, \cos u\sin v, \frac{\sqrt{2}}{2}cos(2u), \sin u\cos v, \sin u\sin v, \frac{\sqrt{2}}{2}\sin(2u)),$$

where $u = \sqrt{\frac{c}{2}}x$, $v = \frac{\sqrt{6c}}{2}y$. f is a pseudo-parallel immersion in $\mathbb{S}_c^5$ with $\Phi = \frac{-c}{2}$. Thus, if $i : \mathbb{S}_c^5 \to \mathbb{S}_c^5 \times \mathbb{R}$ is the totally geodesic inclusion given by $i(x) = (x, 0)$, by Proposition 2 we have that $i \circ f$ is a pseudo-parallel immersion in $\mathbb{S}_c^5 \times \mathbb{R}$ with non vanishing normal curvature.

Question

1. Are there other examples, up to isometries, of pseudo parallel surfaces in $\mathbb{Q}_c^3 \times \mathbb{R}$ $(c \neq 0)$, which T is not a principal direction?
2. Is there an isometric immersion of a topological 2-sphere into $\mathbb{S}^4 \times \mathbb{R}$ that is not included in a slice?

Conjecture:

"The only minimal Φ-pseudo-parallel surfaces in $\mathbb{Q}_c^4 \times \mathbb{R}$ with non vanishing normal curvature and constant Φ are given for $i \circ f$ where i is totally geodesic in $\mathbb{S}_c^4 \times \mathbb{R}$ and f the Veronese surface". See *Conjecture* in [4]"

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