The indeterminacy locus of the Voisin map

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Abstract

Beauville and Donagi proved that the variety of lines $F(Y)$ of a smooth cubic fourfold Y is a hyperkähler variety. Recently, C. Lehn, M.Lehn, Sorger and van Straten proved that one can naturally associate a hyperkähler variety $Z(Y)$ to the variety of twisted cubics on Y. Then, Voisin defined a degree 6 rational map $\psi : F(Y) \times F(Y) \dashrightarrow Z(Y)$. We will show that the indeterminacy locus of ψ is the locus of intersecting lines.

Introduction

It is a classical result that a manifold with a Ricci flat metric has trivial first Chern class, and by Bogomolov's decomposition such manifolds have a finite étale cover given by the product of a Torus, Calabi-Yau varieties and hyperkähler varieties. Hyperkähler manifolds are interesting in their own and they are the subject of an intensive research. The first examples are K3 surfaces, and Beauville proved in that for any $n \geq 0$, the Hilbert schemes of points $X^{[n]}$, where X is a $K3$ surface or the generalized Kummer varieties K^nA associated to an Abelian surface A, are hyperkähler varieties. Any hyperkähler variety that is a deformation of $X^{[n]}$ where X is a $K3$ surface (respectively, a generalized Kummer variety) is called $K3^{[n]}$-type (respectively, K^nA-type). Those examples are particularly interesting because they permit to construct hyperkähler varieties of any even complex dimension. Later, O'Grady constructed two new examples in dimension 6 and 10 of hyperkähler varieties that are not deformation of known types. There are few explicit complete families of hyperkähler of $K3^{[n]}$-type. Beauville and Donagi proved that the variety of lines $F(Y)$ of a smooth cubic fourfold $Y \subseteq \mathbb{P}^5$ is an hyperkähler variety of $K3^{[2]}$-type. Another example was given much more recently by C. Lehn, M. Lehn, Sorger and van Straten in [3]. They observed that if $F_3(Y)$ is a compactification of the space of rational cubic curves in Y, then $F_3(Y)$ is a $\mathbb{P}^2$-fibration based on a smooth variety $Z'(Y)$. Moreover there is a divisor of $Z'(Y)$ that can be contracted and this contraction produces a hyperkähler variety $Z(Y)$. This variety is of $K3^{[4]}$-type by [1].

On the other hand the study of k-cycles on smooth complex projective varieties is a classical subject and it is very interesting on hyperkähler manifolds with respect to several regards. For example, while it is a classical result that the cone of nef divisors is contained in the cone of pseudoeffective divisors, in general $\mathrm{Nef}_k(X) \not\subseteq \overline{\mathrm{Eff}}_k(X)$ for $2 \leq k \leq \dim X - 2$. For example, Ottem proved that if the cubic Y is very general, then the second Chern class $c_2(F(Y))$ of the Fano variety of lines in Y is nef but it is not effective [5].

It is known, due to Mumford's Theorem, that for a projective hyperkähler variety X of dimension $2n$ the kernel of the cycle map $cl : A^{2n}(X) \to H^{4n}(X)$ is infinite-dimensional. Nevertheless, Beauville conjectured that the cycle class map is injective on the subalgebra of $A^*(X)$ generated by divisors. On the other hand Shen and Vial in [6] used a codimension 2 algebraic cycle to give evidence to the existence of a certain decomposition for the Chow ring of $F(Y)$ for a very general cubic fourfold Y.

Voisin constructed in [7] a degree 6 rational map $\psi : F(Y) \times F(Y) \dashrightarrow Z(Y)$. We proved [4] that $\mathrm{Ind}(\psi)$ is the locus of intersecting lines. Later, Chen [2] proved that a blow up of I is a resolution of the indeterminacy of ψ. We hope that the explicit description of $\mathrm{Ind}(\psi)$ will contribute to the study of $c_2(Z(Y))$, the study of algebraic cycles on $Z(Y)$ and to other aspects of the geometry of $Z(Y)$.

Construction of $Z(Y)$

Since $\mathrm{Hilb}^{3z+1}(\mathbb{P}^3) = \mathrm{Hilb}^{gtc}(\mathbb{P}^3) \cup H_1$, where $\mathrm{Hilb}^{gtc}(\mathbb{P}^3)$ is a 12-dimensional smooth component such that the general point is a rational normal curve, and H_1 is a 15-dimensional smooth component such that the general point is a curve C such that C_{red} is a plane cubic. A generalized twisted cubic is a curve with class in $\mathrm{Hilb}^{gtc}(\mathbb{P}^3)$. It is known that $\mathrm{Hilb}^{3z+1}(\mathbb{P}^5)$ contains a smooth irreducible component $\mathrm{Hilb}^{gtc}(\mathbb{P}^5)$ that parameterizes generalised twisted cubics. We define $F_3(Y) := \mathrm{Hilb}^{gtc}(\mathbb{P}^5) \cap \mathrm{Hilb}^{3z+1}(Y)$.

There is a commutative diagram induced by the span map

$$\begin{array}{ccc} F_3(Y) & \longrightarrow & G(4,6) \\ {\scriptstyle \phi}\downarrow & \nearrow_{g} & \\ Z'(Y) & & \end{array}$$

where $Z'(Y)$ is a smooth irreducible projective variety. The diagram has the following remarkable: ϕ is a $\mathbb{P}^2$-fibration and g is finite of degree 72 over the open subset $V_{ADE} \subseteq Z'(Y)$ of surfaces with at most ADE singularities. Moreover, there exists a divisorial contraction $\sigma : Z'(Y) \to Z(Y)$ making $Z'(Y)$ be the blow up of a variety $Z(Y)$ over a subvariety canonically isomorphic to Y. The variety $Z(Y)$ is an hyperkähler.

The Voisin map

There is a degree 6 rational map $\psi : F(Y) \times F(Y) \dashrightarrow Z(Y)$ such that $\psi^*\sigma_Z = pr_1^*\sigma_F - pr_2^*\sigma_F$. Here σ_Z, resp. σ_F denotes the holomorphic 2-form of $Z(Y)$, resp. $F(Y)$. Roughly speaking, the map ψ sends a general pairs of non-incident lines $(l, l') \in F(Y) \times F(Y)$ to the (class of the) degree 3 rational normal curve in the linear system $|L - L' - K_{S_{l,l'}}|$ of the smooth cubic surface $S_{l,l'} := \langle L, L' \rangle \cap Y$. Furthermore, ψ is of maximal rank where it is defined. Finally, there is a subscheme $W \subset F(Y) \times F(Y)$ and a commutative diagram.

$$\begin{array}{ccc} F(Y) \times F(Y) \setminus W & \overset{\rho_{|F \times F \setminus W}}{\dashrightarrow} & G(4,6) \\ {\scriptstyle \psi_{|F \times F \setminus W}}\downarrow & \nearrow_{g_Z} & \\ Z(Y) \setminus Y & & \end{array}$$

Theorem

The indeterminacy locus of ψ is the variety I of the intersecting lines.

Upcoming Research

We hope that using $\mathrm{Ind}(\psi) = I$, we could prove that the cycle $c_2(Z(Y))$ is nef but not effective. This would give a new example of a variety with a nef cycles that is not effective.

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