On Some Symmetric Configurations of the N Body Problem

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Abstract

We considered the restrict problem of the n+1 body, that consist in n equal masses m in the vertices of the regular polygon, a central mass m_0 and other negligible mass μ. We determined the existence and unicity of the positions of Relative Equilibrium for the mass μ in the symmetrical axes $\theta = 0\, mod(\pi/n)$ and for symmetrical axes $\theta = \frac{\pi}{n} mod(2\pi/n)$ with an additional restriction to the value of m_0 given by the values of n. Moreover, by these results we are proofing the finiteness of specific of Central Configurations classes.

Introduction

In 2003, Bang and Elmabsout [1], developed techniques of integral representation of complex functions, with directionally at expression that appear naturally while if analysis the problem of polygonal configurations in the n body problem. Such expression generalizes the expression before developed by Tisserand in [4], and used by Lindow, in [2], in the treatment of a problem equivalent at that been considered by Bang and Elmabsout in [1]. Still in the work [1], Bang and Elmabsout showed a result, of great interest to us, which precisely determines the spacial positions for the negligible mass μ for that the configuration of n equal masses in the vertices of a regular polygon and a central mass m_0 it to be a Central Configuration.

Objectives

Our work consist in to consider the analysis of the existence and unicity of the positions of Relative Equilibrium to μ, using the theory developed in [4] too. We consider the function $\psi(r, \theta)$ that defines the Relative Equilibrium as shown in [1] and we apply direct and indirect analysis methods in search of their zeros. We do a complete demonstration independent of existence and unicity for the two symmetrical axes; $[\theta = 0\, mod(\pi/n)]$ and $[\theta = \frac{\pi}{n} mod(2\pi/n)]$. Moreover, by these results we are proofing the finiteness of specific of Central Configurations classes. What responds with partiality to the sixth problem of Smale [3].

Results

Polygonal Solutions

If we considered n masses m_k in the vertices of a regular polygon. We have that your Mass Center is given by:

$$z_* = \sum_j m_j \omega_j / M,$$

where ω_j are the n-th roots of the unit, and M is the Total Mass. With this notation we have that:

Theorem [Perko-Walter]
If for $n \geq 2$ the functions $z_k(t) = (\omega_j - z_*)e^{i\omega t}$ are solutions of the n Body Problem, then it follows that

$$\omega^2 = \frac{M\gamma}{n},$$

where $\gamma = \frac{1}{4}\sum_{j=1}^{n-1} \csc \frac{\pi j}{n}$.

With this classic result we take possession of very relevant information to sustain our conjectures and start from them for our results. Before, however, we must make a quick foray into the object we are dealing with. That is, the Relative Equilibrium.

Existence of Relative Equilibrium

We considered the equations of the Central Configurations showed in [1]. He has the form:

$$\psi(r,\theta) = m_0(1 - r^3) + n \cdot \left(h_n\left(\frac{1}{r}, \theta\right) - r^3 h_n(1) \right).$$

Bang showed using a theory of the integral representation for complex functions, that the solutions of the equation

$$\psi(r,\theta) = 0,$$

are in the symmetry axes of the polygon. What generalizes the symmetry theorem of Lindow [2].

Results

We consider the function $\psi(r,\theta)$. We restrict our analysis to the plane and following the results of symmetry we wish to be able to make statements about the positions of the Relative Equilibrium. We succeed in this sense when we consider the first axis of symmetry, when we can show that the derivative of $\psi(r, 0)$ is defined as negative, which gives the equation $\psi(r, 0) = 0$ existence and uniqueness of solutions. These solutions are described in the following theorem. There is a bifurcation behavior that we do not address here.

Theorem 1 We considered n masses in vertices of a regular polygon and a negligible mass μ in the symmetrical axes $\theta = 0\, mod\left(\frac{\pi}{n}\right)$. For the position of the negligible mass considered polar coordinates (r, θ). For a central mass $m_0 = 0$, there exist a unique position of Relative Equilibrium to a negligible mass μ, as $r > 1$, and a position $r = 0$. For a central mass $m_0 > 0$, there exist exactly two positions of Relative Equilibrium of the negligible mass μ, one to $0 < r < 1$ and other to $r > 1$.

For the second axis of symmetry, we need conditions derived from an adaptation of the equation: $\psi(r, \frac{\pi}{n}) = 0$ that influences the values of the central mass from fixed values of n. These conditions define a function $G^{(n)}(r)$ that we use to permeate us when the values of n are sufficiently large in comparison with the value of the central mass m_0 so that there are Relative Equilibrium positions.

Theorem 2 Considering n equal masses in vertices of a regular polygon and a negligible mass μ in the symmetrical axes $\theta = \frac{\pi}{n}$, and a central mass $m_0 \geq 0$. For our contents lets going consider to negligible mass μ the polar coordinates (r, θ). Under this considerations has the follow affirmatives:

1. For $m_0 = 0$ and $0 \leq r < 1$ there exist unique position of Relative Equilibrium to negligible mass μ;
2. For $0 < m_0 \leq G^{(n)}(r)$ while $G^{(n)}(r) > 0$ and $0 < r < 1$ there exist two positions of Relative Equilibrium to negligible mass μ;
3. For $m_0 > G^{(n)}(r)$ and $0 < r < 1$ not there exist position of Relative Equilibrium to negligible mass μ;
4. For $m_0 \geq 0$ and $r > 1$ there exist unique position of Relative Equilibrium to negligible mass μ that to be in interval $(1, 1 + \sqrt[3]{4})$, and for $n \geq 21$ to be in interval $\left(1, 1 + \sqrt[3]{\frac{2n}{n+1}}\right)$.

Conclusion

Under all the considerations made on the restricted problem of $n + 1$ bodies. And also with the notation specified throughout the results we can present our conclusion in result format:

Theorem 3 Lets n equal masses in vertices of a regular polygon and one central mass m_0, across around of your center as constant angular velocity ω. Considered the addition of a particle μ, of negligible mass. If $m_0 > G^{(n)}(r)$ then exist exactly $3n$ Central Configurations classes.

References

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