The fractional Schrödinger equation on the half line

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1. Introduction

Presentation of the model

The one dimensional fractional Schrödinger equations

$$i\partial_t u(t,x) + (-\Delta)^{\alpha/2}u(t,x) = \lambda |u(x,t)|^2 u(x,t),\ x,t\in\mathbb{R}$$

was introduced in the theory of the fractional quantum mechanics where the Feynmann path integrals approach is generalized to α-stable Lévy process . Also it appears in the water wave models.
In this presentation, we study the following initial boundary value problem (IBVP) on the positive half-line

$$\begin{cases} i\partial_t u(t,x) + (-\Delta)^{\alpha/2}u(t,x) = \lambda|u(t,x)|^2u(t,x), & (t,x)\in\mathbb{R}\times\mathbb{R}^+,\\ u(0,x)=u_0(x), & x\in\mathbb{R}^+,\\ u(0,t)=f(t), & t\in(0,T), \end{cases} \tag{1}$$

where $\alpha\in(1,2)$ and the nonlocal operator $(-\Delta)^{\alpha/2}$, is defined by

$$(-\Delta)^{\alpha/2}v(x) = \int_{\mathbb{R}} e^{ix\xi}|\xi|^{\alpha/2}\hat{\tilde{v}}(\xi)d\xi \quad \text{for } v\in C_0^\infty(\mathbb{R}^+) \tag{2}$$

and

$$\tilde{v} = \begin{cases} v(x), & \text{for } x\geq 0,\\ v(-x), & \text{for } x<0. \end{cases}$$

where

$$u_0\in H^s(\mathbb{R}^+) \text{ and } f\in H^{\frac{2s-1+\alpha}{2\alpha}}(\mathbb{R}^+). \tag{3}$$

we are interested on the following questions for the IBVP (1):

- *Is the IBVP* (1) *local well-posedness in the low regularity Sobolev space, more precisely, in $H^s(\mathbb{R}^+)$ for $0\leq s<\frac{1}{2}$?*
- *Is there some smoothing effect for the IBVP* (1)*?*

We state the main theorem for IBVP (1) as follows.

Theorem 1.1 *Let $s\in(\frac{2-\alpha}{4},\frac{1}{2})$. For given initial-boundary data u_0 and f satisfying* (3) *there exist a positive time $T := T\left(\|u_0\|_{H^s(\mathbb{R}^+)}, \|f\|_{H^{\frac{2s-1+\alpha}{2\alpha}}(\mathbb{R}^+)}\right)$ and unique solution $u(t,x)\in C((0,T);H^s(\mathbb{R}^+))$ of the IBVP* (1)*, satisfying*

$$u\in C\big(\mathbb{R}^+;\ H^{\frac{2s-1+\alpha}{2\alpha}}(0,T)\big)\cap X^{s,b}((0,T)\times\mathbb{R}^+),$$

for some $b(s)<\frac{1}{2}$. Moreover, the map $(u_0,f)\longmapsto u$ is analytic from $H^s(\mathbb{R}^+)\times H^{\frac{2s-1+\alpha}{8}}(\mathbb{R}^+)$ to $C\big((0,T);\ H^s(\mathbb{R}^+)\big)$.
Moreover, for $a<$ min$\{\frac{\alpha-1}{2},\frac{4s+\alpha-2}{2}\}$ holds

$$u(x,t) - L_{u_0,f}(x,t)\in H^{s+a}(\mathbb{R}^+), \tag{4}$$

for all t in $(0,T)$, where $L_{u_0,f}(x,t)$ denotes the solution of the linearized IBVP (1)*.*

Throughout of this presentation, we fix a cutoff function $\psi\in C_0^\infty(\mathbb{R})$ such that $\psi(t)=1$ if $t\in[-1,1]$ and supp $\psi\subset[-2,2]$.

2. Preliminary

2.1 Bourgain spaces

For $s,b\in\mathbb{R}$, and $\alpha\in(1,2)$ we introduce the classical Bourgain spaces $X^{s,b}$ related to the Schrödinger equation as the completion of $S'(\mathbb{R}^2)$ under the norms

$$\|u\|_{X^{s,b,}} = \left(\int\int \langle\xi\rangle^{2s}\langle\tau-|\xi|^\alpha\rangle^{2b}|\hat{u}(\xi,\tau)|^2 d\xi d\tau\right)^{\frac{1}{2}}$$

2.2 Riemman-Liouville fractional integral

The tempered distribution $\frac{t_+^{\alpha-1}}{\Gamma(\alpha)}$ is defined as a locally integrable function for Re $\alpha>0$ by

$$\left\langle \frac{t_+^{\alpha-1}}{\Gamma(\alpha)}, f\right\rangle = \frac{1}{\Gamma(\alpha)}\int_0^{+\infty} t^{\alpha-1}f(t)dt.$$

For Re $\alpha>0$, we have that

$$\frac{t_+^{\alpha-1}}{\Gamma(\alpha)} = \partial_t^k\left(\frac{t_+^{\alpha+k-1}}{\Gamma(\alpha+k)}\right), \tag{5}$$

for all $k\in\mathbb{N}$.

Definition 2.1 *If $f\in C_0^\infty(\mathbb{R}^+)$, we define*

$$\mathcal{I}_\alpha f = \frac{t_+^{\alpha-1}}{\Gamma(\alpha)} * f.$$

para todo $\alpha\in\mathbb{C}$

3. Linear Version

We define the unitary group associated to the linear fractional Schrodinger equation as

$$v(t) := e^{it(-\Delta)^{\alpha/2}}\phi = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi}e^{it|\xi|^\alpha}\hat{\phi}(\xi)d\xi$$

where $\alpha\in(0,2)$. v is a solution of the following problem

$$\begin{cases} i\partial_t v(t,x) + (-\Delta)^{\alpha/2}v(t,x) = 0, & (t,x)\in\mathbb{R}\times\mathbb{R},\\ v(0,x)=\phi(x), & x\in\mathbb{R}. \end{cases} \tag{6}$$

Lemma 3.1 *Let $s\in\mathbb{R}$. if $\phi\in H^s(\mathbb{R})$, then for $\alpha\in(1,2)$ and $b\in(0,1)$ we have*

$$\|e^{it(-\Delta)^{\alpha/2}}\phi(x)\|_{C(\mathbb{R}_t;H^s(\mathbb{R}_x))} + \|\varphi(t)e^{it(-\Delta)^{\alpha/2}}\phi(x)\|_{C(\mathbb{R}_x;H^{\frac{2s-1+\alpha}{2\alpha}}(\mathbb{R}_t))}$$
$$+\|\psi(t)e^{it(-\Delta)^{\alpha/2}}\phi(x)\|_{X^{s,b}} \leq c\|\psi\|_{H^1(\mathbb{R})}\|\phi\|_{H^s(\mathbb{R})}.$$

4. The Duhamel Boundary Forcing Operator

First, we consider $\alpha\in(1,2)$ and define the function B is defined as follow

$$B(x) = \int_{\mathbb{R}} e^{ix\xi}\ e^{i|\xi|^\alpha}\ d\xi$$

is well defined and the proof is standard.
On the other hand, consider the following equation

$$\begin{cases} i\partial_t u + (-\Delta)^{\alpha/2}u = \dfrac{2\pi}{B(0)\Gamma(1-\frac{1}{\alpha})}\delta_0(x)\,\mathcal{I}_{\frac{1}{\alpha}-1}f(t), & (x,t)\in\mathbb{R}\times(0,T)\\ u(x,0)=0, \quad x\in\mathbb{R} \end{cases} \tag{7}$$

For any f we define

$$\mathcal{L}f(x,t) = \frac{1}{B(0)\Gamma(1-\frac{1}{\alpha})}\int_0^t (t-t')^{-1/\alpha}B\left(\frac{x}{(t-t')^{1/\alpha}}\right)\mathcal{I}_{\frac{1}{\alpha}-1}f(t')\,dt'$$

which is the solution of (7).
Now we state the needed estimates for the Duhamel boundary forcing operators class.

Lemma 4.1 *Let $0<b<\frac{1}{2}$, $-\frac{1}{2}<s<\frac{2\alpha-1}{\alpha}$ and $\alpha\in(1,2)$. Then*

$$\|\mathcal{L}f(x,t)\|_{C\left(\mathbb{R}_t;\,H^s(\mathbb{R}_x)\right)} + \|\psi(t)\mathcal{L}f(x,t)\|_{C\left(\mathbb{R}_x;\,H_0^{\frac{2s+\alpha-1}{2\alpha}}(\mathbb{R}_t^+)\right)}$$
$$+\|\psi(t)\mathcal{L}f(x,t)\|_{X^{s,b}} \leq c\|f\|_{H_0^{\frac{2s+\alpha-1}{2\alpha}}(\mathbb{R}^+)}.$$

5. Nonlinear Versions

We define the Duhamel inhomogeneous solution operator $\mathcal{D}$ as

$$\mathcal{D}w(x,t) = -i\int_0^t e^{i(t-t')(-\Delta)^{\alpha/2}}w(x,t')dt',$$

which is a solution for the following problem

$$\begin{cases} i\partial_t v(x,t) + (-\Delta)^{\alpha/2}v(x,t) = w(x,t),\ (x,t)\in\mathbb{R}\times\mathbb{R},\\ v(x,0)=0,\ x\in\mathbb{R}. \end{cases} \tag{8}$$

Lemma 5.1 *Let $0<s<1/2$ and $-\frac{1}{2}<c\leq 0\leq b\leq c+1$, then:*

$$\|\psi(t)\mathcal{D}w(x,t)\|_{C\left(\mathbb{R}_t;\,H^s(\mathbb{R}_x)\right)} + \|\psi(t)\mathcal{D}w(x,t)\|_{C(\mathbb{R}_x;H^{\frac{2s-1+\alpha}{2\alpha}}(\mathbb{R}_t))}$$
$$+\|\psi(t)\mathcal{D}w(x,t)\|_{X^{s,b}} \leq \|w\|_{X^{s,c}}.$$

6. Bilinear Estimate

Proposition 6.1 *For $\frac{2-\alpha}{4}<s$ and $a<\min\{\frac{\alpha-1}{2},\frac{4s+\alpha-2}{2}\}$ there exist $\epsilon>0$ such that for $\frac{1}{2}-\epsilon<b<\frac{1}{2}$, we have*

$$\||u|^2u\|_{X^{s+a,-b}} \lesssim \|u\|_{X^{s,b}}^3$$

7. Theorema 1.1: Idea of the proof

Let $Z^{s,b}$ the Banach space given by

$$Z^{s,b} = C(\mathbb{R}_t,H^s(\mathbb{R}_x))\cap C(\mathbb{R}_x,H^{\frac{2s-1+\alpha}{2\alpha}}(\mathbb{R}_t))\cap X^{s,b}$$

under the norm

$$\|v\|_{Z^{s,b}} = \sup_{t\in\mathbb{R}}\|v(t,\cdot)\|_{H^s} + \sup_{x\in\mathbb{R}}\|v(\cdot,x)\|_{H^{\frac{2s-1+\alpha}{2\alpha}}} + \|v\|_{X^{s,b}},$$

Let

$$\Lambda(u)(t) = \psi(t)e^{it(-\Delta)^{\alpha/2}}\tilde{u}_0 + \psi(t)\mathcal{D}\left(|u|^2u\right)(t) + \psi(t)\mathcal{L}h(t)$$

where $h(t) = \left(\chi_{(0,+\infty)}\psi(t)f(t) - \psi(t)e^{it(-\Delta)^{\alpha/2}}\,\tilde{u}_0|_{x=0} - \psi(t)\mathcal{D}\left(|u|^2u\right)(t)|_{x=0}\right)\Big|_{(0,+\infty)}$

Our goal is to show that Λ defines a contraction map on any ball $Z^{s,b}$

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