Renormalization of multicritical circle maps

Gabriela Estevez
Instituto de Ciências Matemáticas e de Computação
Universidade de São Paulo - Brazil
gestevez@icmc.usp.br

Abstract

We give necessarily conditions on the renormalizations and on the rotation number to transform a topological conjugacy, between two multicritical circle maps, into a differentiable one. This is joint work with Pablo Guarino (UFF-Brazil).

Introduction

In [10] Yoccoz proved that any two C^3 orientation preserving circle homeomorphisms with non-flat critical points and same irrational rotation number are topologically conjugate. A question that arise is about the conditions to get regularity on the conjugacy, or in other words, to get rigidity.

For maps with only one critical point there are a lot of works directed to answer this question [2], [3], [8], [7], [9]; any two C^4 critical circle map with same irrational rotation number and same odd criticality are conjugate by a C^1 diffeomorphism. If their rotation number belongs to $\mathcal{A}$ (a total Lebesgue measure set), then the conjugacy is a $C^{1+\alpha}$ diffeomorphism.

For maps with more than one critical point, we know [4] that the conjugacy (between maps with same number of critical points) that sends critical point to critical point is quasi-symmetric. Here we prove the following:

Theorem 0.1 (Main Theorem)**.** *[6] Let f and g be two C^r, $r \geq 3$, multicritical circle maps with same number of critical points and rotation number belonging to $\mathcal{A}$. Let h be the conjugacy between f and g that sends critical points into critical points. If the renormalizations of f and g around corresponding critical points converge exponentially in the C^1 topology, then h is a $C^{1+\alpha}$ diffeomorphism.*

Settings

Definition 1. *A multicritical circle map is an orientation-preserving C^3 circle homeomorphism, with irrational rotation number and non-flat critical points. That is, for each $c_i \in Crit(f)$ there exists a neighborhood W_i of c_i, and an integer $d_i > 1$ such that $f(x) = f(c_i) + \phi_i(x)\left|\phi_i(x)\right|^{d_i - 1}$ for all $x \in W_i$, where $\phi_i : W_i \to \phi(W_i)$ is a C^3 diffeomorphism with $\phi(c_i) = 0$. The number d_i is called the criticality of c_i.*

Example 1. ***Generalized Arnold's family***

$$f_{a,b}(x) = x + a - \left(\frac{b}{2N\pi}\right)\sin(2N\pi x)\ (mod\ 1),\quad a \in [0,1),\ b \geq 0 \text{ and } N \in \mathbb{N},$$

If $b = 1$, $f_{a,1}$ has N critical points, all of them with same criticality equal to three.

For $\rho \in [0,1) \setminus \mathbb{Q}$, we consider its truncated continued fraction sequence:

$$\frac{p_n}{q_n} = [a_0, a_1, \cdots, a_{n-1}] = \cfrac{1}{a_0 + \cfrac{1}{\ddots \cfrac{1}{a_{n-1}}}}.$$

The sequence $\{q_n\}_{n\in\mathbb{N}}$ satisfies: $q_0 = 1$, $q_1 = a_0$, and for $n \geq 1$ $q_{n+1} = a_n q_n + q_{n-1}$.

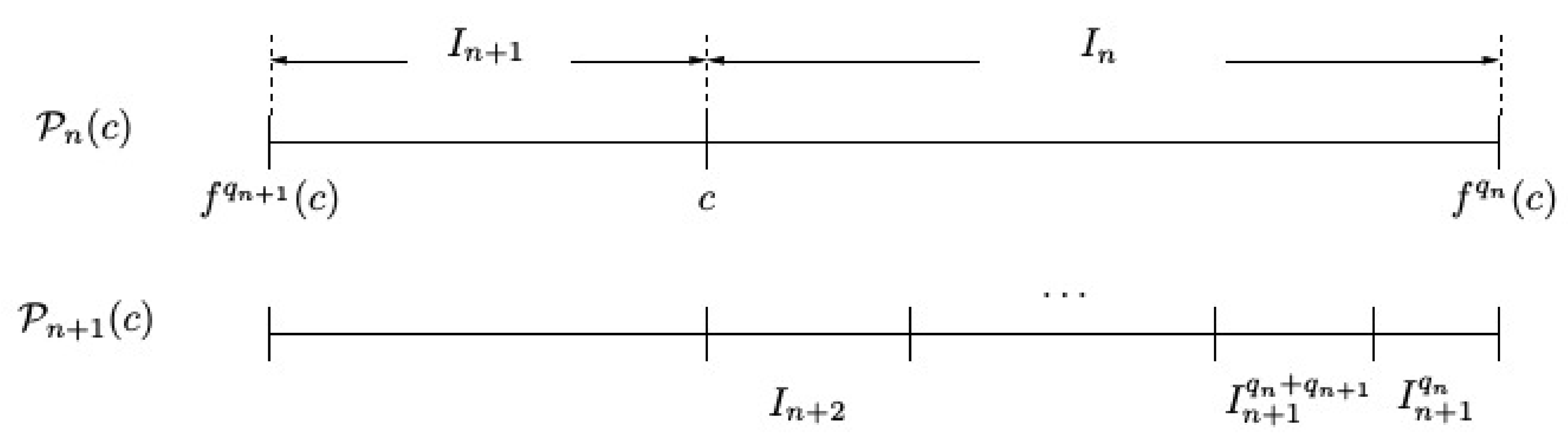
Let f be a multicritical circle map and let $x \in S^1$. The closed interval with endpoints x and $f^{q_n}(x)$, containing the point $f^{q_{n+2}}(x)$, contains no other iterate $f^j(x)$ for $1 \leq j \leq q_n - 1$.

Let $I_n(x)$ be the interval with endpoints x and $f^{q_n}(x)$ containing $f^{q_{n+2}}(x)$. For each $n \geq 0$ and each $x \in S^1$, the collection of intervals

$$\mathcal{P}_n(x) = \left\{f^i(I_n(x)) : 0 \leq i \leq q_{n+1} - 1\right\} \bigcup \left\{f^j(I_{n+1}(x)) : 0 \leq j \leq q_n - 1\right\}$$

is a circle partition (module endpoints) called the *n-th dynamical partition* of f associated to the point x.

We note that the partition $\mathcal{P}_{n+1}(x)$ is a (non-strict) refinement of $\mathcal{P}_n(x)$:



We have the following result concerning dynamical partitions associated to a critical point:

Theorem 0.2 (Beau Bounds)**.** *[5] Given $N \in \mathbb{N}$ and $d > 1$, let $\mathcal{F}_{N,d}$ be the family of all multicritical circle maps with N critical points whose maximum criticality is d. There exist $n_0 \in \mathbb{N}$ and $C_B = C_B(N,d) > 1$ such that, for every $f \in \mathcal{F}_{N,d}$, each $c \in Crit(f)$, any $n \geq n_0$ and every adjacent pair of intervals $I, J \in \mathcal{P}_n(c)$, we have*

$$\frac{1}{C_B} \leq \frac{|I|}{|J|} \leq C_B.$$

Comments on the Proof

To simplify the notation, we assume that f only has two critical points, namely c_0 and c_1.

Definition 2. *Let $\mathcal{A} \subset [0,1]$ be the set of rotation numbers $\rho = [a_0, a_1, \cdots]$ satisfying:*

1) $\limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \log a_j < \infty$.

2) $\lim_{n\to\infty} \frac{1}{n} \log a_n = 0$.

3) $\frac{1}{n}\sum_{j=k+1}^{k+n} \log a_j \leq \omega\left(\frac{n}{k}\right)$*, for all $0 < n \leq k$, where ω is a map which only depends on the rotation number such that $\omega(t) > 0$ for all $t > 0$, and $t\omega(t) \to 0$ as $t \to 0$.*

By [2], A has total Lebesgue measure.

Definition 3. *A fine grid is a sequence $\{\mathcal{Q}_n\}_{n\geq 0}$ of finite circle partitions such that:*

(a) Each $\mathcal{Q}_{n+1}$ is a strict refinement of $\mathcal{Q}_n$;

(b) There exists an integer $b \geq 2$ such that each atom $\Delta \in \mathcal{Q}_n$ is the disjoint union of at most b atoms of $\mathcal{Q}_{n+1}$;

(c) There exists $\tilde{C} > 1$ such that $\tilde{C}^{-1}|\Delta| \leq |\Delta'| \leq \tilde{C}|\Delta|$ for each pair of adjacent atoms $\Delta, \Delta' \in \mathcal{Q}_n$.

Proposition 1. *[1] Let h be a circle homeomorphism such that for each pair of adjacent intervals $I, J \in \mathcal{Q}_n$ and for all $n \geq 0$, there exist a constant $C > 0$ and $\lambda \in (0,1)$ satisfying*

$$\left|\frac{|I|}{|J|} - \frac{|h(I)|}{|h(J)|}\right| \leq C\,\lambda^n. \tag{1}$$

Then h is a $C^{1+\alpha}$-diffeomorphism.

So, we need to construct a fine grid satisfying inequality (1) to get our Main Theorem:

1) We define a partition $\widehat{\mathcal{P}}_n$ such that their vertices in $J_n(c_k) = I_n(c_k) \cup I_{n+1}(c_k)$, for $k \in \{0,1\}$, are iterates of first return of c_k or of the pre-image of the other critical point.

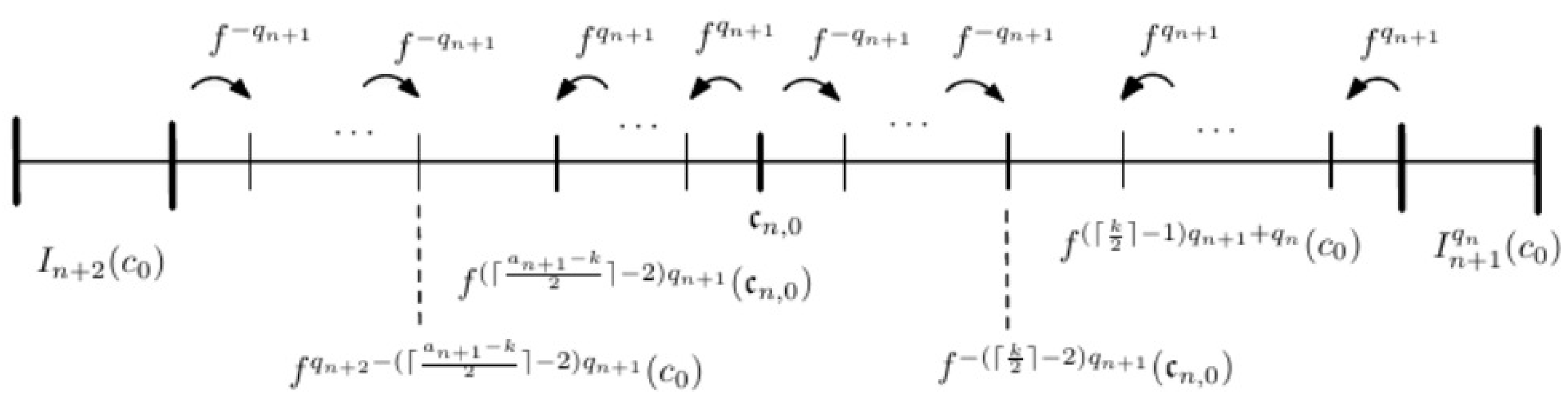


Figure 1: The new partition in $J_n(c_0)$, where $\mathfrak{c}_{n,0}$ is the pre-image of c_1.

2) We prove the following:

Lemma 0.1 (Key Lemma)**.** *There exists $\tilde{C} > 0$, $\tilde{K} > 1$ and $\mu^* \in (0,1)$ such that given $n, p \in \mathbb{N}$ with $n \geq n_0$ and ν be a vertex of $\widehat{\mathcal{P}}_{n+p}$, we have*

a) If $\nu \in J_n(c_0)$ then

$$|\nu - h(\nu)| \leq \tilde{C}\,\tilde{K}^p\,|J_n(c_0)|\,(\mu^*)^n.$$

b) If $\nu \in J_n(c_1)$ then

$$|\nu - h(\nu)| \leq \tilde{C}\,\tilde{K}^p\,|J_n(c_1)|\,(\mu^*)^n.$$

3) We join some intervals in $J_n(c_k)$, in a suitable way, and then we spread this union around the circle, to obtain a fine grid.

4) From 2) and 3) we can obtain inequality (1) for vertices in $J_n(c_0)$ and in $J_n(c_1)$. Using the Koebe principle we get inequality (1) for the rest of the vertices.

References

[1] L. Carleson. On mappings, conformal at the boundary. *J. Anal. Math.*, 19(1):1–13, 1967.

[2] E. de Faria and W. de Melo. Rigidity of critical circle mappings i. *J. Eur. Math. Soc.*, 1(4):339–392, 1999.

[3] E. de Faria and W. de Melo. Rigidity of critical circle mappings ii. *J. Amer. Math. Soc.*, 13(2):343–370, 2000.

[4] G. Estevez and E. de Faria. Real bounds and quasisymmetric rigidity of multicritical circle maps. *Trans. Amer. Math. Soc.*, 370(8):5583–5616, 2018.

[5] G. Estevez, E. de Faria, and P. Guarino. Beau bounds for multicritical circle maps. *Indagationes Math.*, 29:842–859, 2018.

[6] G. Estevez and P. Guarino. Renormalization of multicritical circle maps. *Work in progress*.

[7] P. Guarino, M. Martens, and W. de Melo. Rigidity of critical circle maps. *Duke Math. J.*, 167(11):2125–2188, 2018.

[8] K. Khanin and A. Teplinsky. Robust rigidity for circle diffeomorphisms with singularities. *Invent. Math.*, 169(1):193–218, 2007.

[9] D. Khmelev and M. Yampolsky. The rigidity problem for analytic critical circle maps. *Mosc. Math. J*, 6(2):317–351, 2006.

[10] J-C. Yoccoz. Il nya pas de contre-exemple de denjoy analytique. *CR Acad. Sci. Paris Sér. I Math*, 298(7):141–144, 1984.