Ekeland's variational principle and Lions' Lemma applied to a problem in $\mathbb{R}^N$

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Abstract

A general variational framework is stabilished for problems involving the Laplacian Operator in $\mathbb{R}^N$. The well-known equation

$$-\Delta u + u = f(u), u \in H^1(\mathbb{R}^N) \tag{1}$$

is studied with suitable hypothesis on $f : \mathbb{R} \to \mathbb{R}$ using Ekeland's Variational Principle and bypassing the lack of compactness with Lions' Lemma, proving the existence of a weak solution to the problem.

Introduction

This work aims in studying variational methods for proving the existence of weak solutions to some elliptic Partial Differential Equations. As Sobolev's continuous and compact immersions form the basic arsenal of such framework, problems in unlimited domains in $\mathbb{R}^N$ present additional difficulties due to the lack of Sobolev Compact Immersions, and thus lead to the need of additional tools (like Lions' Lemma) for stabilishing non-triviality of the constructed solution.

As such, this work begins by stabilising the variational framework for studying a partial differential equation in $\mathbb{R}^N$ involving the Laplacian operator, followed by Ekeland's Variational Principle and Lions' Lemma, and ends with the application of those propositions showing the existence of a weak solution to that problem.

Variational Framework

The variational method consists in finding a functional I such that searching for weak solutions for the original problem is reduced to finding critial points of I. For the elliptic problem considered, the functional and it's Fréchet derivative are given by (for all $v \in H^1(\mathbb{R}^N)$)

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{1}{2}\int_{\mathbb{R}^N} |u|^2 - \int_{\mathbb{R}^N} F(u) \tag{2}$$

$$I'(u)v = \int_{\mathbb{R}^N} \nabla u \nabla v + \int_{\mathbb{R}^N} uv - \int_{\mathbb{R}^N} f(u)v \tag{3}$$

By Sobolev's Continuous Immersions: given a domain $\Omega \subseteq \mathbb{R}^N$, $H^1(\Omega) \hookrightarrow L^s(\Omega)\, \forall s \in [2, 2^*]$, where $2^* = \begin{cases} \frac{2N}{N-2}, \text{ if } N \geq 3 \\ \infty, \text{ if } N \in \{1,2\} \end{cases}$. As a consequence, there exists $C > 0$ such that $|u|_s \leq C\|u\|_{H^1(\Omega)}$ for all $u \in H^1(\Omega)$.

Obs: if Ω is bounded, H^1 is changed to H^1_0.

Fundamental Theorems

Theorem 1 (Ekeland's Variational Principle): Let X be a complete metric space and $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ a lower-semicontinuous functional, $\not\equiv +\infty$, bounded from below. Let $\epsilon, \lambda > 0$ and $u \in X$ be such that $\Phi(u) \leq \inf_X \Phi + \frac{\epsilon}{2}$. Then, there exists $v_\epsilon \in X$ such that **(a)** $\Phi(v_\epsilon) \leq \Phi(u)$; **(b)** $d(v_\epsilon, u) \leq 1/\lambda$; **(c)** $\Phi(v_\epsilon) < \Phi(w) + \epsilon\lambda d(v_\epsilon, w)\, \forall w \in X \setminus v_\epsilon$.

Proof key idea: Define the equivalence relation $w \prec v \iff \Phi(w) \leq \Phi(v) - \epsilon d(w,v)$ in X and the subset sequence $(A_n)_{n=1}^{\infty}$ recurservely as: $u_0 = u, A_0 = \{w \in X : w \prec u_0\}$, $u_n \in X$ such that $\Phi(u_n) \leq \inf_{A_{n-1}} \Phi + 1/n$ and $A_n = \{w \in X : w \prec u_n\}$ for $n \geq 1$. All A_n are closed, $A_n \supset A_{n+1}$ for all n and the diameter of A_n approaches zero as n approaches infinity, so the intersection of all such sets contains only one element, v_ϵ, which satisfies (a), (b) and (c).

Corolary 1: Let E be a Banach Space and $I : E \to \mathbb{R}$ a lower-semicontinuous funcional bounded from below. Suppose I is Frechét differentiable for all $u \in E$, then given $\delta > 0$, there exists $u_\delta \in E$ such that $I(u_\delta) \leq \inf_{u \in E} I(u) + \frac{\delta}{2}$ and $\|I'(u_\delta)\|_{E'} \leq \delta$.

Theorem 2 (Lions' Lemma): Let $r > 0$ and $q \in [2, 2^*)$. If $(u_n) \subset H^1(\mathbb{R}^N)$ is limited in $H^1(\mathbb{R}^N)$ and if $\sup_{y \in \mathbb{R}^N} \int_{B_r(y)} |u|^q \to 0$, then $u_n \to 0$ in $L^s(R^N)$ for all $s \in (2, 2^*)$.

Proof key idea: Apply Sobolev and Hölder inequalities and cover the space with balls of radius r such that each point of the space is contained in at most $N+1$ balls.

Corolary 2: If (u_n) is a bounded $(PS)_c$ sequence for I, then only one of the following items occur: **(a)** the sequence converges to zero in $H^1(\mathbb{R}^N)$; **(b)** there exists $(y_n) \subset \mathbb{R}^N$ and $r, \beta > 0$ such that

$$\int_{B_r(y_n)} |u|^2 \geq \beta > 0 \tag{4}$$

Non-trivial weak solution

Theorem 3: If $f \in C^0(\mathbb{R}, \mathbb{R})$ satisfies: (f_0) $f(s) = 0\, \forall s \leq 0$; (f_1) $\lim_{s \to 0} f(s)/s = 0$; (f_2) $\limsup_{s \to \infty} f'(s)/|s|^{q-2} < +\infty$; (f_3) $s \to f(s)/s$ is increasing $\forall s > 0$; and (f_4) there exists $\theta > 2$ such that $0 < \theta F(s) \leq sf(s)\, \forall s > 0$, then (1) has a weak solution.

Proof key ideas: The conditions above gives us estimates on f and F. Restricting I to the Nehari Manifold, we apply Corolary 1 and get a candidate sequence. By standard arguments we prove convergence (up to a subsequence) to a critical point u. Using Corolary 2, we define $v_n(x) = u(x + y_n)$, which weakly converges to a non-trivial critical point v, a weak solution.

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