Alexandroff Maximum Principle and priori estimates for solutions to quasilinear equations.

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Resumo

We discuss one kind of maximum principle, Alexandroff gives the estimates in terms of the L^n- norm.
In what follows we will prove a sequence of a priori estimates, which get sharper and sharper, and build on each other. The root estimate, was obtained first by Alexandrov in the 1950's. Important contributions, which are not so easy to distinguish from Alexandrov's work, are due to Bakelman; and later this estimate appeared in a work of C. Pucci. Thus, at least in the Western literature, this estimate became known as the Alexandrov-Bakelman-Pucci estimate, to which we will refer from now on as the ABP estimate. It is interesting to observe that the ABP estimate is not so much a result of diferential equations, but rather of convex analysis and measure theory.

Introdução

Let Ω be an open connected set in $\mathbf{R}^n$ Let L be the second order diferential operator:

$$L = a_{ij}(x)D_{ij} + b_i(x)D_i + c(x) \tag{1}$$

with $a_{ij}(x) \in \mathrm{L}^{\infty}_{loc}(\Omega)$ and $b_i, c \in \mathrm{L}^{\infty}(\Omega)$. Without loss of generality one assumes $a_{ij} = a_{ji}$.

For an elliptic operator L as in (1) one defines

$$D^*(x) = det(a_{ij}(x))^{\frac{1}{n}}$$

Indeed, the definition of ellipticity implies that D^* is well-defined, $D^*(x)$ is the geometric average of the eigenvalues of $a_{ij}(x)$ and even that $0 \leq \lambda \leq D^* \leq \Lambda$, where $\lambda = \min a_{ij}(x)$ and $\Lambda = \max a_{ij}(x)$.

Theorem 1

Suppose $u \in \mathrm{C}(\overline{\Omega}) \cup \mathrm{C}^2(\Omega)$ satisfies $\mathrm{L}u \geq$ in Ω with the following conditions:

$$\frac{|b|}{D^*}, \frac{f}{D^*} \in \mathrm{L}^n(\Omega) \quad \text{and} \quad c \leq 0 \quad \text{in} \quad \Omega.$$

then there holds

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u + C\left|\left|\frac{f^-}{D^*}\right|\right|_{\mathrm{L}^n(\Gamma^+)}.$$

where Γ^+ is the upper contact set of u and C is a constant depending only on n, diam(Ω) and $\left|\left|\frac{b}{D^*}\right|\right|_{\mathrm{L}^n(\Gamma^+)}$.

Lemma 2

Suppose $g \in \mathrm{L}^1_{loc}(\mathrm{R}^n)$ is nonnegative. Then for any $u \in \mathrm{C}(\overline{\Omega}) \cup \mathrm{C}^2(\Omega)$ there holds

$$\int_{B_M(0)} g \leq \int_{\Gamma^+} g(Du)|\mathrm{det}D^2u|$$

where Γ^+ is the upper contact set of u and $M = (\sup_{\Omega} u - \sup_{\partial\Omega} u)/d$, wich $d = diam(\Omega)$.

Remark 3

For any positive definite matrix $A = a_{ij}$ we have

$$det(-D^2u) \leq \left(\frac{-AD_{ij}u}{nD^*}\right)^n \quad \text{on} \quad \Gamma^+$$

In the following we use The Theorem to derive some a priori estimates for solutions to quasi-linear equations.

Application 1

Suppose $u \in \mathrm{C}(\overline{\Omega}) \cup \mathrm{C}^2(\Omega)$ satisfies the mean curvature equation,

$$(1+|Du|^2)\triangle u - D_i u D_j u D_{ij} = nH(x)(1+|Du|^2)^{\frac{3}{2}} \quad in \quad \Omega.$$

for some $H \in \mathrm{C}(\Omega)$. Then if

$$H_0 \equiv \int_{\Omega} |H(x)|^n dx < \omega_n$$

we have

$$\sup_{\Omega} |u| \leq \sup_{\partial\Omega} |u| + C\,diam(\Omega)$$

Prova
We have

$$a_{ij}(x,z,p) = (1+|p|^2)\delta_{ij} - p_i p_j$$

$$b_i(x,z,p) = -nH(x)(1+|p|^2)^{\frac{3}{2}}$$

$$D = \left(1+|p|^2 - p_1^2\right)\left(1+|p|^2\right)^{n-2}\left(1+|p|^2 - \sum_{i=2}^{n} p_i^2\right) - \left(1+|p|^2\right)^{n-2}\sum_{i=2}^{n} p_i^2 p_1^2 = \left(1+|p|^2\right)^{n-1}$$

This implies

$$\frac{|b_i(x,z,p)|}{nD^*} \leq \frac{n|H(x)|\left(1+|p|^2\right)^{\frac{3}{2}}}{n\left(1+|p|^2\right)^{\frac{n-1}{n}}} = \frac{|H(x)|}{\left(1+|p|^2\right)^{\frac{-(n+2)}{2n}}} = \frac{h(x)}{g(p)}$$

$$\int_{\mathrm{R}^n} g^n(p)dp = \int_{\mathrm{R}^n}\left(1+|p|^2\right)^{\frac{-(n+2)}{2}} = \omega_n$$

$$\int_{\Omega} h^n(x)dx = \int_{\Omega} |H(x)|^n \leq \omega_n$$

We may apply Lemma 2 to g^n and get

$$\int_{B_M(0)} g^n \leq \int_{\Gamma^+} g^n(p)|\mathrm{det}D^2u| \leq \int_{\Gamma^+} g^n(p)\left(\frac{b}{nD^*}\right)^n = \int_{\Gamma^+} h^n(x)$$

Therefore there exists a positive constant C, depending only on g and h, such that $M \leq C$
The second case is about the prescribed Gaussian curvature equation.

Corollary 3 Suppose $u \in \mathrm{C}(\overline{\Omega}) \cup \mathrm{C}^2(\Omega)$ satisfies,

$$det(D^2u) = K(x)(1+|Du|^2)^{\frac{n+2}{2}} \, in \, \Omega.$$

for some $K \in \mathrm{C}(\Omega)$. Then if

$$K_0 \equiv \int_{\Omega} |K(x)|^n dx < \omega_n$$

we have

$$\sup_{\Omega} |u| \leq \sup_{\partial\Omega} |u| + C\,diam(\Omega)$$

Referências

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