The Cachy Problem for Generalized Fractional Semilinear Type Plate Equation

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Abstract. In this work we study the existence and uniqueness of solutions for a second order semilinear δ-α-γ-evolution equation with fractional damping term which includes some partial differential equations as plate equation under effects of rotational inertia or not and Boussinesq type that can model hydrodynamics problems.

Introduction. We consider in this work the following Cauchy problem for a second order generalized δ-α-γ-evolution equation with a fractional damping in $\mathbb{R}^n$,

$$\begin{cases} \partial_t^2 u + (-\Delta)^\delta \partial_t^2 u + b\Delta^2 u \\ \quad + a(-\Delta)^\alpha u + (-\Delta)^\theta \partial_t u = \beta(-\Delta)^\gamma u^p, \\ u(0,\,x) = u_0(x), \quad \partial_t u(0,\,x) = u_1(x), \end{cases} \tag{1}$$

where $u = u(t,\,x)$ with $(t,\,x) \in\,]0,\ \infty[\times\mathbb{R}^n$, $b,\,a > 0$, $\beta \in \mathbb{R}$ constants, $p > 1$ a integer and u_0, u_1 are the initial data. The exponents δ, α, θ and γ of the Laplacian operator are such that $0 \leq \delta \leq 2$, $0 \leq \alpha \leq 2$, $0 \leq \theta \leq (2+\delta)/2$ and $0 \leq \gamma \leq (2+\delta)/2$.

The function $u = u(x,\,t)$, for example, in the case $\delta = 1$ and $\beta = 0$, describes the transverse displacement of a plate under to effects of rotational inertia and without non-linear effects and a fractional dissipation represented by the term $(-\Delta)^\theta \partial_t u$. In the case $\delta = 0$, $\beta \neq 0$ and $\gamma = 1/2$ the differential equation in (1) models the nonlinear displacement of the plate without rotational inertial effects.

Problem Linear Description. Through the Theory of Semigroup we show the existence and uniqueness of solutions to the following Cauchy problem

$$\begin{cases} \partial_t^2 u + (-\Delta)^\delta \partial_t^2 u + \Delta^2 u + a(-\Delta)^\alpha u \\ \quad + (-\Delta)^\theta \partial_t u = 0 \\ u(0,\,x) = u_0(x), \quad \partial_t u(0,\,x) = u_1(x), \end{cases} \tag{2}$$

where $u = u(t,\,x)$, with $(t,\,x) \in\,]0,\ \infty[\times\mathbb{R}^n$ and $a > 0$ is a constant. The exponents of Laplacian δ, α, θ are such that $0 \leq \delta \leq 2$, $0 \leq \alpha \leq 2$ and $0 \leq \theta \leq (2+\delta)/2$.

We consider two cases depending on the exponents of the Laplacian operators and for each case we consider operators that reduced the problem to a first order one. The energy space $X = H^2(\mathbb{R}^n) \times H^\delta(\mathbb{R}^n)$ let us rewrite the problem (2) in matrix form

$$\frac{dU}{dt} = BU + J(U), \quad U(0) = U_0 \quad \text{for all} \quad t > 0.$$

where $U = (u,\,\partial_t u)$, $U(0) = (u_0,\,u_1)$ and the operators B and J adequate for each case.

Using the Lumer Phillips Theorem, we prove that B is the infinitesimal generator of contraction semigroup of class C_0 in X and that J is a bounded operator in X, that is, exist only one solution for the Cauchy Problem (2).

Theorem 1.1. *Let $n \geq 1$, $0 \leq \delta \leq 2$ and $0 \leq \theta \leq (2+\delta)/2$. If $u_0 \in H^{4-\delta}(\mathbb{R}^n)$ e $u_1 \in H^2(\mathbb{R}^n)$ then the Cauchy Problem (2) have only one solution u in following class*

$$C^2([0,\ \infty[; H^\delta(\mathbb{R}^n)) \cap C^1([0,\ \infty[;\ H^2(\mathbb{R}^n)) \\ \cap C\left([0,\ \infty[; H^{4-\delta}(\mathbb{R}^n)\right).$$

Problem Semilinear Description. We consider the Cauchy problem for a semilinear equation

$$\begin{cases} \partial_t^2 u + (-\Delta)^\delta \partial_t^2 u + \Delta^2 u + a(-\Delta)^\alpha u \\ \quad + (-\Delta)^\theta \partial_t u = \beta(-\Delta)^\gamma u^p, \\ u(0,\,x) = u_0(x), \quad \partial_t u(0,\,x) = u_1(x), \end{cases} \tag{3}$$

where $u = u(t,\,x)$, with $(t,\,x) \in\,]0,\ \infty[\times\mathbb{R}^n$, $a > 0$, $\beta \neq 0$ and $p > 1$ integer. The fractional powers of the Laplacian operator are considered as follows $0 \leq \delta \leq 2$, $0 \leq \alpha \leq 2$, $0 \leq \theta \leq (2+\delta)/2$ and $1/2 \leq \gamma \leq (\alpha+\delta)/2$.

As in the linear problem, to study the existence of solutions, we need to consider two cases:

$$\text{i) } 0 \leq \theta < \delta;\ 0 \leq \delta \leq 2 \quad \text{ii) } 0 \leq \delta \leq \theta;\ 0 \leq \theta \leq \frac{2+\delta}{2}.$$

We reduce the order of the Cauchy Problem (3) and rewrite it in the following matrix form

$$\frac{dU}{dt} = BU + F(U), \quad U(0) = U_0 \quad \text{for all} \quad t > 0$$

where $U = (u,\,\partial_t u)$, $U_0 = (u_0,\,u_1)$ and the operator B is define in the Linear Problem of according to each both cases mentioned above and it is the infinitesimal generator of a contractions semigroup of class C_0 in X. The operator F is the operator which contains the non-linear term.

Global Existence. We show for the semilinear problem that the maximal interval of existence in the two previous cases is $[0,\ \infty[$, by assuming the additional condition that $\gamma \geq \alpha/2$. For this, let us assume that $T = T_m < \infty$ and we will prove that $\|U\|_X + \|BU\|_X < \infty$, so we conclude that $T_m = \infty$.

To show that the solution of the Cauchy Problem (3) is global, that is, $T = T_m = +\infty$ we will need the following elementary lemma

Lemma 1.2. *Let $p > 1$ and $F(M) = aI_0 + bT^2M^p - M$, $M \geq 0$, a continuous positive function, with a, b, I_0, T positive constants and $M \geq 0$. Then, there exists unique $M_0 > 0$ absolute minimum point of $F(M)$ in $[0,\ \infty[$. In addition, there exists $\varepsilon > 0$ such that if $0 < I_0 \leq \varepsilon$ then $F(M_0) < 0$.*

Therefore, if $0 < I_0 \leq \varepsilon$ and $\varepsilon > 0$ given by Lemma 1.2, due to the continuity of the function $M_1(t)$, there are only two possibilities,

i) $M_1(t) < M_0\ \forall\, t \in [0,\ T_m[$ or **ii)** $M_1(t) > M_0\ \forall\, t \in [0,\ T_m[$.

Then, assuming another condition on the initial data that $M_1(0) < M_0$ (M_0 the global minimum point of Lemma 1.2), it follows that $M_1(t) \leq M_0$ for all $t \in [0,\ T_m[$ by the continuity of $M_1(t)$ with $t \in [0,\ T_m]$. The condition **i)** holds.

Theorem 1.3. *Let $0 \leq \theta < \delta$, $0 \leq \delta \leq 2$, $\alpha/2 \leq \gamma \leq (\alpha+\delta)/2$, $p > 1$ integer and $1 \leq n < 8 - 2\delta$. Consider the initial data $u_0 \in H^{4-\delta}(\mathbb{R}^n)$ and $u_1 \in H^2(\mathbb{R}^n) \cap \dot{W}_{-\alpha,2}(\mathbb{R}^n)$ satisfying $0 < I_0 \leq \varepsilon$ and $M_1(0) < M_0$ with ε, I_0, M_0, $M_1(0)$ given above and in Lemma 1.2. Then there exists unique global solution $u = u(t,\,x)$ for Cauchy Problem (3) in the class*

$$u \in C^2([0,\ \infty[;\ H^\delta(\mathbb{R}^n)) \cap C^1([0,\ \infty[;\ H^2(\mathbb{R}^n)) \\ \cap C([0,\ \infty[;\ H^{4-\delta}(\mathbb{R}^n)).$$

References

[1] J. L. Horbach, R. Ikehata, & R. C. Charão, Optimal Decay Rates and Asymptotic Profile for the Plate Equation with Structural Damping, *Journal of Mathematical Analysis and Applications*, v. 440, n. 2, p. 529-560, 2016.