Well-posed Quasi-linear Parabolic System with Applications

Eduardo Arbieto Alarcon
IME/UFG
arbietoalarcon@ufg.br

Abstract. The purpose of this work is to present a unified treatment of the Cauchy problem (local and global in time) for various quasi-linear partial differential equations the "parabolic type".

The abstract equation of evolution we consider has the form

$$\frac{du}{dt} + A(t,\, u)\, u = f(t,\, u) \in X, \tag{1}$$

where X and Y are Banach Spaces, $A(t,\, y)$ is a linear operator from Y into X for each $(t,\, y) \in [0,\, T_0] \times W$, W an open subset of Y and $f\colon [0,\, T_0] \times W \to X$.

In application we usually have $Y \hookrightarrow X$, i.e., Y is densely and continuously embedded in X and the Cauchy problem.

Introduction. In 1961, in article "*Abstract evolution equations of parabolic type in Banach and Hilbert spaces*". Nagoya J. Math. 19, T. Kato constructs the *evolution operator* (or *fundamental solution*) $U(t,\, s)$, defined for $0 \le s \le t \le T$, such that any *strict solution* de (L) (i.e. a function $u(t)$ such that $u(t)$ is strongly continuous on $[0,\, T]$, strongly continuously differentiable in $(0,\, T]$ and (L) is satisfied for $t \in (0,\, T]$), can be expressed in the form

$$u(t) = U(t, 0)\, \phi + \int_0^t U(t,\, s)\, f(s)\, ds. \tag{2}$$

It shows the exists a unique evolution operator $U(t,\, s) \in \mathcal{B}(X)$ defined for $0 \le s \le t \le T$, with the following properties. $U(t,\, s)$ is strongly continuous for $0 \le s \le t \le T$ and

$$U(t,\, r) = U(t,\, s)\, U(s,\, r), \quad r \le s \le t,$$
$$U(t,\, t) = I.$$

For $s < t$, **the range of** $U(t,\, s)$ **is a subset of** $\mathrm{Dom}[A(t)]$ and

$$A(t)\, U(t,\, s) \in \mathcal{B}(X), \quad \|A(t)\, U(t,\, s)\| \le M|t - s|^{-1},$$

where M is a constant depending only on θ, h, k, T, M_0, M_1 and M_3. Furthermore, $U(t,\, s)$ is strongly continuously differentiable in s for $t > s$ and

$$\frac{\partial}{\partial t} U(t,\, s) + A(t)\, U(t,\, s) = 0.$$

See too Kato T., Tanabe H.,"*On the abstract evolution equation*", Osaka Math J. 14 (1962). In this article T. Kato assumes that $-A(t)$ generates a semigroup $e^{-s\, A(t)}$ which is holomorphic in a sector containing the positive s-axis and for some $h = 1/m$, where m is a positive integer, $\mathrm{Dom}[A(t)^h] = \mathcal{D}$ is independent of t.

In 1970,Tosio Kato in "*Linear evolution equations of "hyperbolic" type.*" J. Fac. Sci. Univ. Tokyo Sect. I 17 (1970),241-258., weakens the assumption of $D(A(t))$ constant, introducing the conditions: "stability", "A-admissible", existence of a Banach space Y is densely and continuously embedded in X, such that $Y \subset D(A(t))$, existence of a family $\{S(t)\}$ of isomorphisms of Y onto X such that

$$S(t)\, A(t)\, S(t)^{-1} = A(t) + B(t),$$

The extension to the quasi-linear case is given in Tosio Kato "*Quasi-linear equations of evolution, with applications to partial differential equations.* Spectral theory and differential equations (Proc. Sympos., Dundee, 1974; pp. 25-70. Lecture Notes in Math., Vol. 448, Springer, Berlin, (1975); with applications to different kinds of quasi-linear differential equations such as symmetric hyperbole systems of the first order, wave equations, Korteweg-de Vries equation, Navier-Stokes and Euler equation, equations for compressible fluids, magnetohydrodynamic equations, coupled Maxwell and Dirac equations, etc.

Kato's idea is simple and elegant. For each v in a convenient metric space space, solve the linear Cauchy Problem

$$(L^v) \quad \frac{du}{dt} + A(t,\, v)\, u = f(t,\, v) \in X, \quad u(0) = \phi \in Y \tag{3}$$

to obtain a map $v \longmapsto \Phi(v) = u_v$ and show that this map is a contraction.

Problem Quasi-linear Description. Next, we study the initial value problem for a model(see [Mota-Schecter] (*Combustion fronts in a porous medium with two layers*, Journal of Dynamics and Differential Equations, **18** (3), (2006), 615-665) for the lateral propagation of a combustion front through a porous medium with two parallel layers having different properties. The reaction involves oxygen and a solid fuel. In each layer, the model consist of a nonlinear reaction-diffusion-convection system, derived from balance equations and Darcy's law. Under an incompressibility assumption, we obtain a simple model whose variables are temperature and unburned fuel concentration in each layer, denoted, respectively, by u_i and y_i, i = 1,2, as state variables depending on (x, t). The model includes heat transfer between the layers, and has the form:

$$\begin{cases} (u_i)_t - \dfrac{\lambda_i}{a_i + b_i y_i}(u_i)_{xx} + \dfrac{c_i}{a_i + b_i y_i}(u_i)_x = \\ = \dfrac{b_i A_i u_i + d_i}{a_i + b_i y_i} y_i f(u_i) + (-1)^i q \dfrac{u_1 - u_2}{a_i + b_i y_i}, \\ (y_i)_t = -A_i y_i f(u_i), \\ (u_i(x, 0), y_i(x, 0)) = (u_{i,0}(x), y_{i,0}(x)), \end{cases} \tag{4}$$

When $y_i(x)$ is a known function of (x, t), we have a semi-linear system

$$\begin{cases} (u_i)_t - \frac{\lambda_i(x)}{a_i(x) + b_i(x) y_i(x,t)}(u_i)_{xx} + \frac{c_i(x)}{a_i(x) + b_i(x) y_i(x,t)}(u_i)_x = \\ f_i(x, t, u), \ u_1(x, 0) = \phi_1(x), \quad u_2(x, 0) = \phi_2(x), \\ x \in R, \quad t > 0, \quad i = 1, 2, \quad u = (u_1, u_2), \end{cases} \tag{5}$$

We prove existence and uniqueness of local solutions for the initial value problem (5) in $C\left([0,\, T]; H^2(\mathbb{R})\right) \cap C^1\left([0,\, T]; L^2(\mathbb{R})\right)$, we use the method by T. Kato in *Abstract differential equations and nonlinear mixed problems*. Accademia Nazionale Dei Lincei, Scuola Normale Superiore. Lezioni Fermiane. Pisa, Italy, (1985); in case $\mathrm{Dom}[A(t)]$ is independent of t and (5) is a CD-systems in Kato's sense (see [Theorem 1.2.]). Now in the quasi-linear case (unknown $y_i(x,\, t)$) we use Kato's results in *Quasilinear equations of evolution, with applications to partial differential equations*, Lect. Note in Math., 448, (1975), to solve the local Cauchy problem. For this we solve the following quasi-linear problem:

$$\frac{dv}{dt} + A(v)\, v = 0 \in X, \ v(0) \in Y, \tag{6}$$

where $X = L^2(\mathbb{R}) \times L^2(\mathbb{R})$, $Y = H^2(\mathbb{R}) \times L^2(\mathbb{R})$ and W subset of X, $v = (v_1,\, v_2) \in W$, $A_1(v)(\phi,\, \psi) = -p(v_2)\, \phi'' + q(v_2)\, \phi'$, $A_2(v)(\phi,\, \psi) = r(v_1)\, \psi$. We assume that $A(v)$ is uniformly parabolic and its coefficients are bounded and smooth in $\mathbb{R}$, we obtain the following result

Theorem 1.1. *Let* $v(0) \in W = \{(v_1,\, v_2)\colon ||v_1||_{H^2(\mathbb{R})} < r_1,\ ||v_2||_{H^{s_0}(\mathbb{R})} < r_2\}$, $s_0 > 5/2$. *Then is a unique solution* v *to (6) such that* $v \in C\left([0, T], W\right)$.

Global Existence. To obtain global existence for the semi-linear case (5), we must combines global a *priori* estimates for $||u||_Y$ with the following extension principle: Let $\phi = u(0) \in Y$ and assume that (5) is locally well-posed. Let $T^*(\phi) = \sup\{T > 0\colon \exists!\ \text{solution of}(5)\ \text{in}\ [0,\, T]\}$. Then, either $T^*(\phi) = \infty$ or $T^*(\phi) < \infty$ and $\lim_{t \uparrow T^*(\phi)} ||(u(t),\, y(t))|| = \infty$, (see Iorio, Rafael José, Jr.; Iorio, Valéria de Magalhães *Fourier analysis and partial differential equations*. Cambridge Studies in Advanced Mathematics, 70. Cambridge University Press, Cambridge, 2001. xii+411 pp. To obtain a priori estimates we need regularity properties typical of the parabolic equations, for this we prove that $B(t) = A\left(v(t)\right)$ is m-sectorial in space X in the sense of T. Kato, and we also prove that the family the operator $\{B(t)\colon t \ge 0\}$ satisfies the condition of theorem the T. Kato ([1]), obtaining the regularity sufficient for local solution. Then we show that exists unique global solution $u = u(t,\, x)$ for Cauchy Problem (5) in the class

$$u \in C\left([0, +\infty), Y\right) \cap C^1\left([0, +\infty), X\right).$$

The next work is to extend the results to the quasi-linear case, preserving the regularity of the parabolic type of the local solution, to obtain a priori estimates and thus solve the global Cauchy problem.

References

[1] T. Kato -*Abstract evolution equations of parabolic type in Banach and Hilbert spaces*. Nagoya J. Math. 19(1961), 93-125.