DIFFUSIVE RADIATIVE TRANSFER IN HALF-SPACE

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Resume

In this work we study the Radiative transfer equation in the forward-peaked regime on the Half-Space. More precisely, it is shown that the equation is well-posed by proving instantaneous regularization of weak solutions for arbitrary initial datum in L^1. Classical techniques for hypo-elliptic operators, such as averaging lemma, are used in the argument. Throughout this work we will use the stereographic projection and one gives an expression for the scattering operator given in terms of a fractional Laplace-Beltrami operator on the sphere, or equivalently, a weighted fractional Laplacian analog in the projected plane.

Introduction

Radiative transfer is a physical phenomenon of energy transfer in the form of electromagnetic radiation. The propagation of radiation through a medium is described by absorption, emission, and scattering processes. In the case that the medium is only by scattering process, the radiative transfer equation (RTE) reduces to

$$\begin{cases} \partial_t u + \theta\cdot\nabla_x u = \mathcal{I}(u), & \text{in} \quad (0,T)\times\mathbb{R}^d_+\times\mathbb{S}^{d-1}, \\ u = u_0, & \text{on} \quad \{t=0\}\times\mathbb{R}^d_+\times\mathbb{S}^{d-1}, \\ u = g, \quad \text{on} \quad (0,T)\times\partial\mathbb{R}^d_+\times\mathbb{S}^{d-1} & \text{and} -(v\cdot n(x))>0. \end{cases} \tag{0.1}$$

where T is any arbitrary time and $u = u(t,x,\theta)$ is understood as the radiation distribution in the half-space $(0,T)\times\mathbb{R}^d_+\times\mathbb{S}^{d-1}$. The initial radiation distribution u_0 is assumed to be non negative and $u_o\in L_1(\mathbb{R}^d_+\times\mathbb{S}^{d-1})$. Similarly, we assume that $g(t,x,\theta)$ is also non negative on $(0,T)\times\partial\mathbb{R}^d_+\times\mathbb{S}^{d-1}$ and $-(v\cdot n(x))>0$ and $n(x)$ is the outward unit normal onto the boundary of the domain R^d_+. The scattering operator reads as

$$\mathcal{I}(u) := \mathcal{I}_{b_s}(u) = \int_{\mathbb{S}^{d-1}} (u(\theta') - u(\theta)) b_s(\theta,\theta')\,\mathrm{d}\theta'.$$

We also assume that the angular scattering kernel b_s satisfies the normalized integrability condition such that

$$1 = \int_{\mathbb{S}^{d-1}} b_s(\theta,\theta')\,\mathrm{d}\theta' = \int_{\mathbb{S}^{d-1}} b_s(\theta',\theta)\,\mathrm{d}\theta'.$$

The angular scattering kernel is formally approximated by

$$b_s(\theta,\theta') = \frac{b(\theta\cdot\theta')}{(1-\theta\cdot\theta')^{\frac{d-1}{2}} + s}, \; s\in\left(0,\min\left\{1,\frac{d-1}{2}\right\}\right)$$

where $b(z)\geq 0$ has some regularity in the neighboorhood of $z=1$.

Basic properties of the Scattering Operator

The Jacobian of the stereographic projection and its inverse can be computed respectively as

$$\mathrm{d}v = \frac{\mathrm{d}\theta}{(1-\theta_d)^{d-1}} \qquad \text{and} \qquad \mathrm{d}\theta = \frac{2^{d-1}\,\mathrm{d}v}{\langle v\rangle^{2(d-1)}}.$$

Using the shorthanded notation $\theta = \mathcal{J}(v)$ and $\theta' = \mathcal{J}(v')$ we obtain that

$$1 - \theta\cdot\theta' = \frac{2|v-v'|^2}{\langle v\rangle^2\langle v'\rangle^2}.$$

Goals

1. Obtain an expression for the scattering operator.
2. Prove that actual density function u enjoys higher regularity in both angular and spatial variable.

Results

Proposition 1. *Define $\mathcal{I}_{b_s} = \mathcal{I}_{b(1)} + \mathcal{I}_h$. Then, for any sufficiently regular function u in the sphere the stereographic projection of the operator $\mathcal{I}_{b(1)}$ is given by*

$$\frac{\left[\mathcal{I}_{b(1)}\right]_{\mathcal{J}}}{\langle\cdot\rangle^{d-1}} = \frac{2^{\frac{d-1}{2}-s}b(1)}{c_{d-1,s}}\langle v\rangle^{2s}\left(c_{d,s}\frac{u_{\mathcal{J}}}{\langle\cdot\rangle^{d-1+2s}} - (-\Delta_v)^s w_{\mathcal{J}}\right)$$

where $u_{\mathcal{J}} = u\circ\mathcal{J}$ and $w_{\mathcal{J}} = \frac{u_{\mathcal{J}}}{\langle v\rangle^{d-1+2s}}$. In particular, we have

$$\begin{aligned} &\frac{1}{b(1)}\int_{\mathbb{S}^{d-1}} \mathcal{I}_{b(1)}(u)(\theta)\overline{u(\theta)}\mathrm{d}\theta \\ &\quad = -c_{d,s}\|(-\Delta_v)^{s/2}w_{\mathcal{J}}\|^2_{L^2(\mathbb{R}^{d-1})} + C_{d,s}\|u\|^2_{L^2(\mathbb{S}^{d-1})}. \end{aligned} \tag{0.2}$$

for some explicit positive constants $c_{d,s}$ and $C_{d,s}$. Furthermore, defining the differential operator $(\Delta_\theta)^s$ acting on functions defined on the sphere by the formula

$$[(-\Delta_\theta)^s u]_{\mathcal{J}} := \langle\cdot\rangle^{d-1+2s}(-\Delta_v)^s w_{\mathcal{J}},$$

the scattering operator simply writes as the sum of a singular and a $L^2_\theta-$ bounded parts

$$\mathcal{I}_{b_s} = -D(-\Delta_\theta)^s + c_{s,d}I + \mathcal{I}_h,$$

where $D = 2^{\frac{d-1}{2}-s}\frac{b(1)}{c_{d-1,s}}$ is the diffusion parameter.

Theorem 2. *For any dimension $d\geq 3$ fixed and assume that $u\in C([t_0,t_1]; L^2(\mathbb{R}^d_+,\mathbb{S}^{d-1})$ solves the RTE on the half-space for $t\in(t_0,t_1)$ and $g\in L^2([t_0,t_1)\times\mathbb{R}^{d-1}\times\mathbb{S}^{d-1}$. Then for any $s\in(0,1)$, there exists a constant $C := C(d,s)$ independent of time such that*

$$\begin{aligned} \|(-\Delta_x)^{s_0/2}u\|_{L^2_{t,x,v}} &\leq C\Big(\|u(t_0)\|_{L^2_{x,v}} + \|u\|_{L^2_{t,x,v}} \\ &\leq + \|(-\Delta_v)^s w_{\mathcal{J}}\|_{L^2_{t,x,v}} + \|g\|_{L^2([t_0,t_1)\times\mathbb{R}^{d-1}\times\mathbb{S}^{d-1})}\Big) \end{aligned}$$

where $s_0 = \frac{s/4}{2s+1}$.

Conclusion

- As in the whole space we obtained extra regularity in the space variable, but we can see that it is estimated with an extra term, this term is the boundary condition.
- The above Theorem is so important to prove uniqueness and existence of solutions from the Radiative transfer equation.

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Acknowledgments

E. C. has been supported by CAPES and PETROBRAS postgraduate grants.