Critical Points and Topology: a beautiful love affair

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Abstract

In this poster we investigate how the topology of sublevels of continuously differentiable functionals is related to the existence of critical points. The Palais-Smale condition, the Deformation Lemma and the Minimax Principle are discussed for a functional restricted to a manifold of codimension 1 in a Banach space. The main reference is [1].

Introduction

Most (if not all) laws of nature can be stated as minimization problems, *i.e.* in the form of **variational principles**. The subject of the Calculus of Variations, which was discovered (invented?) in the 18th century as a tool for the study of simple (but nonetheless difficult) problems in mechanics, has evolved in the 20th century and today, together with Topology and Functional Analysis, constitutes the standard framework to deal with nonlinear elliptic PDEs, besides being a fascinating subject in itself.

Variational methods, or minimax principles, are concerned with the existence of critical points of functionals. They are given as the minimization over a given class of sets of the supremum of the functional in these sets. The most important feature, what "makes the theorem work", is the fact that a change in topology when passing from a sublevel to another is related to the existence of a critical level between them.

Some examples

In this poster, let X be a Banach Space and $J : X \longrightarrow \mathbb{R}$ a C^1 functional. The Fréchet derivative of J is denoted by J'. Given $c \in \mathbb{R}$, J^c denotes the sublevel $\{x \in X; J(x) \leq c\}$. The sets J_c and so on are defined analogously.

Example 1. Consider the real functional $J(x) = x^3 - 3x$. It is easily seen that the number of connected components of the sublevels changes when we pass a critical level. □

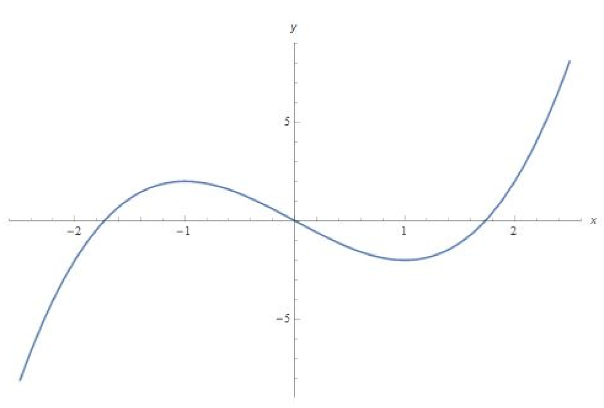

Figure 1: Graph of $J(x) = x^3 - 3x$

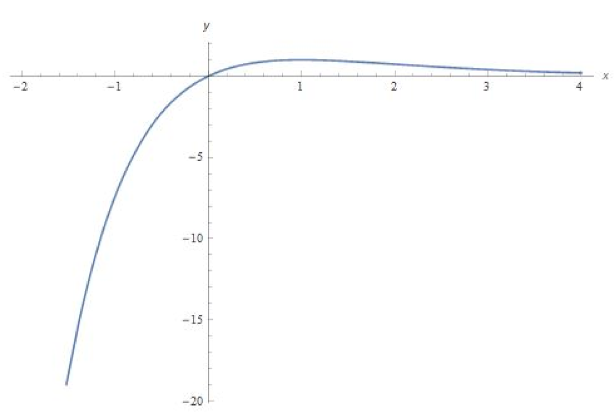

Figure 2: Graph of $J(x) = xe^{1-x}$

Now, a change in the topology of sublevels, alone, is not enough to assert the existence of critical points - we need some compactness:

Example 2. Consider the real functional $J(x) = xe^{1-x}$. We have that $J'(x) = (1-x)e^{1-x}$, so J has a (maximum) critical point at $x = 1$, with critical value $c = 1$. We also have that $\lim_{x\to+\infty} J(x) = 0$. Then, although $\{0\}$ is not a critical level, for any $0 < a < c$ the sublevel J^a has two connected components, while for $a < 0$ the sublevel J^a is an interval. □

Constrained funtionals

For a motivation on why to consider the case of constrained functionals, we begin with an example. Let $\Omega \subset \mathbb{R}^N$ be a regular open set, for $N \geq 3$ and let $p \in (2, 2^*)$. Consider the following problem:

$$(*) \begin{cases} -\Delta u = |u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega \end{cases}$$

One can see this as the problem of minimizing the functional $J(u) = \int_\Omega |\nabla u|^2$ restricted to the unit ball in $L^p(\Omega)$ (for details, see [1]).

Definition. Let X be a Banach space. A **constraint** is a set $S = \{v \in X;\ F(v) = 0\}$ where $F \in C^1(X, \mathbb{R})$ is such that $F'(v) \neq 0$ for all $v \in S$.

Definition. Let $J \in C^1(X, \mathbb{R})$ be a functional and let S be a constraint defined by F. A **constrained critical point** is a point $u \in S$ such that

$$J'(u)[v] = 0 \quad \forall v \in T_u S.$$

Equivalently, u is such that $J'(u) = \lambda F'(u)$. The real number λ is called **Lagrange multiplier**.

A weaker but very useful compactness condition is the following.

Definition. Let X be a Banach space, $J \in C^1(X, \mathbb{R})$ and S a constraint defined by the map F. Then $J|_S$ satisfies the Palais-Smale condition (PS) if every sequence (u_n) such that

$$J(u_n) \to c, \qquad J|_S'(u_n) \to 0$$

has a converging subsequence.

The Deformation Lemma and its consequences

Theorem 1 (Deformation Lemma). *Let X be a Banach space, $F \in C^{1,1}_{loc}(X, \mathbb{R})$ defining the constraint S, $J \in C^1(X, \mathbb{R})$ such that $J|_S$ satisfies PS, $J|_S$ is not constant and $c \in \mathbb{R}$ is not a critical value for $J|_S$. Then there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ there exists a map $\eta \in C(\mathbb{R} \times S, S)$ such that*

(i) *$\eta(0, u) = u$ for all $u \in S$;*

(ii) *If $u \notin J|_{S_{c-\varepsilon_0}}^{c+\varepsilon_0}$ then $\eta(t, u) = u$ for all $t \in \mathbb{R}$;*

(iii) *For each $t \in \mathbb{R}$ $\eta(t, \cdot\,)$ is a homeomorphism from S onto S;*

(iv) *For all $u \in S$ the function $t \mapsto J|_S\big(\eta(t, u)\big)$ is decreasing;*

(v) *$\eta(1, J|_S^{c+\varepsilon}) \subset J|_S^{c-\varepsilon}$;*

(vi) *If $J|_S$ and F are even then $\eta(t, \cdot\,)$ is odd.*

For the proof of the lemma one needs at hand the directions in which the functional J decreases fast enough. In Hilbert spaces, the direction of fastest decrease is given by the gradient, but in a Banach spaces setting the gradient is not available. To overcome this difficulty one introduces a notion of gradient that has just enough to prove what we need. This is the tangent pseudo-gradient vector field.

Let X be a Banach space, S be a constraint defined by F, $J \in C^1(X, \mathbb{R})$ and $x \in S$. Denote the norm of J in the tangent space to S at x by

$$||J'(x)||_* := \sup\left\{J'(x)[y];\ y \in X,\ ||y|| = 1,\ F'(x)[y] = 0\right\}.$$

Also, Let $\tilde{S}$ denote the set of critical points of $J|_S$.

Definition. Let X be a Banach space, $J \in C^1(X, \mathbb{R})$ and S a constraint defined by F. A **tangent pseudo-gradient vector field** for J at u is a locally Lipschitz vector field $v : \tilde{S} \longrightarrow X$ such that for all $u \in \tilde{S}$ the following conditions hold:

$$\begin{cases} ||v(u)|| \leq 2||J'(u)||_*; \\ J'(u)[v(u)] \geq ||J'(u)||_*^2 \quad \text{if} \quad F'(u)[v(u)] = 0. \end{cases}$$

Proposition 1. *Let X be a Banach space, $J \in C^1(X, \mathbb{R})$, $F \in C^{1,1}_{loc}(X, \mathbb{R})$ and S a constraint defined by F. Suppose J is not constant on S. Then there exists a tangent pseudo-gradient vector field v on $\tilde{S}$ that is locally Lipschitz in a neighborhood V_S of S.*

The notion of two sublevels having the same topology is formalized by the following concept.

Definition. Let X be a topological space and $E \subset F$ be two subsets. E is said to be a **deformation retract** of F if there exists a map $\varphi \in C([0, 1] \times F, E)$ such that

$$\varphi(0, \cdot\,) = \text{Id}, \qquad \varphi(t, \cdot\,)\big|_E = \text{Id} \quad \forall t \in [0, 1], \qquad \varphi(1, F) = E.$$

Theorem 2. *Let X be a Banach space, S a constraint defined by $F \in C^{1,1}_{loc}$ and $J \in C^1(X, \mathbb{R})$ such that $J|_S$ is not constant. Then for all regular value c there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the sublevel $J^{c-\varepsilon}$ is a deformation retract of $J^{c+\varepsilon}$.*

Deformations can also be used to find critical levels of functionals. This is important in results regarding mutliplicity of solutions of PDEs.

Theorem 3 (Minimax Principle). *Let X, S, F and J as above. Suppose $J|_S$ satisfies PS. Let $\mathcal{B}$ be a nonempty family of nonempty subsets of X. Suppose that for each $c \in \mathbb{R}$ and $\varepsilon > 0$ the flow η given by the Deformation Lemma is such that $\eta(1, A) \in \mathcal{B} \quad \forall A \in \mathcal{B}$. Define*

$$\tilde{c} = \inf_{A \in \mathcal{B}} \sup_{v \in A} J(v).$$

If $\tilde{c} \in \mathbb{R}$ then $\tilde{c}$ is a critical value of J.

References

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